BEAM Solvers

Congratulations to the successful solvers from Challenge Set 6!

Top Solvers (4+ Solutions, win a prize!)

- Aamirah Spencer
- Alex Torres
- BJ Kemp
- Cindy Truong
- Eils Thornton
- Gittan Johnson
- Ilana Ocampo
- Jacob Tom-Wong
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- Jeremiah Beaufils
- Mohammad Alhusaini
- Nicole Belisario
- Rawin Hidalgo

Solvers (2+ Solutions)

- Camia King Cupid
- Fatima Soussi
- Ivory Wang
- Maryam Yaqoob
- Melvin Chino-Hernandez
- Yilin Li
Challenge Set 6 Solutions

Problem 1

Draw paths to connect A to A, B to B, and C to C, but do it without crossing paths or leaving the box.

(Hint: the answer is NOT to just draw on the edge of the rectangle very carefully. That counts as leaving the box.)

Problem 1 Solution

One thing to notice is that if you draw a line connecting B to B, it will divide the rectangle into two sections. Since the lines we draw can’t cross each other, the two As will need to be on the same side of that line connecting the Bs (and the same is true of the two Cs).

So we need to connect the Bs so that the two As are on the same side; the only way to do this is to go around the bottom of the A box – once we’ve done that, we can connect the As, like this:

Now, we could just connect the blue line to the other B, but if we did that, we wouldn’t be able to draw a line between the two C boxes. So how can we avoid dividing those two? Well, we also go around the bottom C box!
Once we've done that, we can also add a line connecting the two C boxes as well, like this:

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**Problem 2**

If you write out all the numbers from 1 to 300, in how many of the numbers will there be an 8? (Note that 88 only counts as one number with an 8!)

**Problem 2 Solution**

There are two ways a number between 1 and 300 can have an 8 in it:

1. **There is an 8 in the ones digit.** For example, we want to make sure to count 8, 18, 28, and so on. There’s one number like this for every 10, so we need to count how many 10s there are between 1 and 300. We do that by dividing 300 by 10, which gives us 30.

2. **There is an 8 in the tens digits.** For example, we want to make sure to count 80, 81, 82, 83, all the way up to 89. That’s 10 numbers. But there’s also the 10 numbers starting with 180, and the 10 numbers starting with 280. So there are 30 numbers with an 8 in the tens digits.

There’s one other thing we need to consider: are there any numbers of which both things are true? In this case, there are three of them: 88, 188, and 288. So we’re **double counting 3 numbers.** To make sure we don’t do that, we can subtract off the amount we’re double counting at the end: so the total is $30 + 30 - 3 = 57$.

(This is an example of something called the *inclusion-exclusion principle*, and is useful in other situations where we want to count thing that are in two groups that might have some overlap.)
Problem 3

In the previous challenge set, you learned that a palindrome is a whole number that reads the same backwards and forwards. For example, 383, 2882, and 1508051 are all palindromes. What is the difference between the largest six-digit palindrome and the smallest five-digit palindrome?

Problem 3 Solution

First let's find the largest six-digit palindrome and the smallest five-digit palindrome. The largest six-digit palindrome is easy to find because when we make a number big we just use all nines:

999,999.

When we make a number really small, we want to put zeros anywhere we can. But for a number to be five-digits, it has to have a non-zero number in the ten-thousands place (the fifth place over). The smallest non-zero number is 1, so we know the number has to be:

1 _ _ _ _

Since it has to be a palindrome, we know that means the ones digits is also a 1:

1 _ _ _ 1.

We can add zeros everywhere else, to give us:

10001.

Great, so now we just subtract 999,999-10,001 to get the answer 989,998.

Problem 4

The whole numbers 1-100 are written in the table at right in the pattern shown. What is the sum of all the numbers in column E?

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>10 19 18 17 16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 12 13 14 15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 9 8 7 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

:=:=:=:=:=
Problem 4 Solution

There's a lot of good ways to solve the problem that use lots of different methods! We'll go over one, but there's lots of other ways to solve it.

First, I claim that every column adds to the same thing. How can we see this? Let's start by looking at just the first two rows:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
10 & 9 & 8 & 7 & 6
\end{array}
\]

If we add each pair of numbers in the same column here, we always get 11: For example, \(1 + 10 = 11\), \(2 + 9 = 11\), and so on.

This continues to be true as we add more pairs of rows, not just the first and the second: if we add the numbers in each column, they will add to the same thing (but not always 11). This is because each row is exactly 10 more than the row two before it.

This means that if we look at the whole table, the sum of all numbers in column A is the same as the sum of all numbers in column B, and the same with columns C, D and E. So we can add up all the numbers in the table and divide by 5.

What is the sum of the entire table? It's the sum of all numbers up to 100:

\[
1 + 2 + 3 + \ldots + 98 + 99 + 100
\]

Well, we can add this by pairing numbers that add to the same thing:

\[
\begin{align*}
1 + 100 &= 101 \\
2 + 99 &= 101 \\
3 + 98 &= 101 \\
&\vdots
\end{align*}
\]

This pattern continues since the first numbers are going up by exactly 1, and the second ones are going down by exactly 1, so they should always add to the same thing.

How many pairs do we get like this? Well if we keep going all down, we get \(50 + 51 = 101\), so we'll have 50 pairs that add to 101. So this gives us:

\[
50 \times 101 = 5050.
\]

To get the sum of just one column, we need to divide this by 5, so we get 1010 as the answer.
Problem 5
You are allowed to take as many whole numbers as you want that add up to 20, and then multiply them together. What’s the biggest possible result you can get? (This question is special. Finding any result that is bigger than 1000 is worth one point. Finding the highest possible number is worth a bonus point!)

Problem 5 Solution
A good observation that helps with this problem is to notice that taking lots of small numbers usually gives you a bigger product. For example, if you think about $3 + 3 = 6$, multiplying $3 \times 3 = 9$, which is bigger than 6. This is because multiplication makes things bigger than addition.

An exception to this is that even though $2 + 2 + 2 = 6 = 3 + 3$, it’s actually the case that $2^3 < 3^2$. This will be important in a minute!

So we might think that the best thing to do is to only use 2s, since those are the smallest number (other than 1). We get 20 by adding ten 2s together, so then we’d look at:

\[ 2^{10} = 1024 \]

This is over 1000, but not actually the biggest number we can make. Because $2^3 < 3^2$, we can replace groups of three 2s with two 3s and it will give us something bigger! So we keep doing that until we can’t anymore and that gives us six 3s and one singular 2:

\[ 2 + 3 + 3 + 3 + 3 + 3 + 3 = 2 + 6 \times 3 = 2 + 18 = 20 \]

When we multiply them all together, we get:

\[ 2 \times 3^6 = 2 \times 729 = 1458. \]

Problem 6
At BEAM, there were 47 Calderón students and 49 Granville students. Imagine all 96 students stood hand-in-hand in a circle facing into the center. If exactly 23 Calderón students gave their right hand to a Granville student, how many Calderón students gave their left hand to a Granville student?

Problem 6 Solution
We want to figure out how many Calderón students give their left hand to Granville students, but that’s the same number as the number of Granville students that gave the right hand to a Calderón student! So how do we figure that out?
Well, since exactly 23 Calderón students gave their right hand to a Granville student, that means exactly 23 Granville students gave their left hand to a Calderón student. The rest of the Granville students gave their left hand to a Granville student! So there are exactly 49-23=26 Granville students that gave their left hand to a Granville student.

This also means that there are 26 Granville students that gave their right hand to a Granville student. That means the rest of them gave the right hand to a Calderón student: and there are 23=49-26 Granville students left. So 23 Granville students gave the right hand to a Calderón student. Thus there are 23 Calderón students that gave their left hand to a Granville student.

**Problem 7**

In the diagram below, there are many ways to get from A to B by following just the segments and moving only up and to the right.

- For example, marked in red below are three paths that can get you from A to B:

- There are many more paths like this. How many in total? (Hint: there was another problem like this on Challenge Set 4. The solution to that one might help you, because there are too many to list them all out!)
Problem 7 Solution

You'll notice this is a lot like Problem 5 on Challenge Set 4. But this time, there are way too many paths to list them all out! However, we can still use the things we learned from Solution 3.

It works exactly the same way as in Challenge Set 4: at each corner, we write the total number of paths that go to that point, which we can get by adding the number. When that's done, we get:

```
    1  7  28  84  210  462  924
  1  6 21  56 126 252  462
 1  5  15 35  70 126  210
 1  4 10 20  35  56  84
 1  3  6 10 15  21  28
 1  2  3  4  5  6   7
0  1  1  1  1  1  1  1
```

There's also a way to use Solution 2 from Challenge Set 4, but it uses some fancy ways of counting: To get to B, we need to go up exactly 6 times and right exactly 6 times. So we can think of each path as a sequence of U’s (for up) and R’s (for right). However, we still don’t want to write them all out.

Now there are 12 possible moves total we need to make, and we need to choose 6 of them to be ups. Those of you who took Jacob’s class, might recognize this as a combination. We won’t explain why, but you can figure this is:

\[
\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 924.
\]

Problem 1 is by Paul Zeitz. Problem 4 is from MATHCOUNTS. Problem 6 is from Math Kangaroo.