Leontief Production Functions

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Studying vertical foreign direct investment requires a model in which production can be broken up into stages. In this note, we will study the properties of the Leontief production function, which we will use later to model vertical FDI.

The Leontief production function is named after Wassily Leontief, a Nobel Prize winning economist who pioneered input-output analysis.[1] The central assumption of the Leontief production function is that production requires a fixed proportion of inputs. What do we mean by a "fixed proportion of inputs?" Let's look at a few examples.

1. An economics class requires 1 instructor, 1 class room, and 1 group of enrolled students.
2. An automobile requires 4 tires, 1 motor, and 1 steering wheel.
3. An iPhone requires 1 battery, 1 A8 processor, and 1 touch screen.

In each of these examples, production needs exactly the right proportion of inputs to produce the product. If there are too few inputs the product cannot be produced (A car with 3 tires?) and if there are too many inputs, those extra inputs are unused (What would I do with an extra classroom?).

Modeling fixed proportions production

Take, for example, production of a good that requires skilled and unskilled labor in fixed proportions. Mathematically, we can model this production function as

$$x = \min\left\{\frac{\ell_u}{\theta_u}, \frac{\ell_s}{\theta_s}\right\}, \tag{1}$$

where ℓ_s is the hours of skilled labor hired, ℓ_u is the hours of unskilled labor hired, and θ_u and θ_s are the unit input requirements. The *unit input requirements* are numbers that describe how many units of each input are needed to produce one unit of output. The *min* function says that output, x, is equal to the minimum of ℓ_u/θ_u and ℓ_s/θ_s.

Some examples:

1. Let $\theta_u = 2$ and $\theta_s = 1$. Hiring 2 hours of unskilled labor and 1 hour of skilled labor yields 1 unit of output. None of the inputs are wasted.
2. Let $\theta_u = 2$ and $\theta_s = 1$. Hiring 3 hours of unskilled labor and 1 hour of skilled labor yields 1 unit of output. 1 hour of unskilled labor is wasted.
3. Let $\theta_u = 2$ and $\theta_s = 1$. Hiring 1 hour of unskilled labor and 1 hour of skilled labor yields 0.5 unit of output. 0.5 hours of skilled labor are wasted.
4. Let $\theta_u = 2$ and $\theta_s = 1$. Hiring 4 hours of unskilled labor and 2 hour of skilled labor yields 2 unit of outputs. None of the inputs are wasted.

[1] The input-output framework developed by Leontief provides the fundamental organizational structure for economic data.

In this example, the fixed proportion is 2 units of unskilled labor to 1 unit of skilled labor. In examples 2 and 3 above, the factor proportions were not 2-to-1, so some of the input was wasted. In example 4, the ratio of inputs is the correct 2-to-1. Using twice as many inputs in correct proportion produces twice as much output. This means that the production function has *constant returns to scale*.

Numerical example. Consider the production of two different goods, a and b. The production of these goods follows (1), but with different unit labor requirements: $\theta_{ua} = 5$ and $\theta_{sa} = 1$ for good a and $\theta_{ub} = 1$ and $\theta_{sb} = 10$ for good b.

How many hours of skilled and unskilled labor need to be hired to produce 5 units of good a? 5 units of good b? Which good is skilled-labor intensive?

For good a: 25 hours of unskilled labor and 5 hours of skilled labor. For good b: 5 hours of unskilled labor and 50 hours of skilled labor.

Good b is skilled-labor intensive. It requires 10 hours of skilled labor for each hour of unskilled labor, while good a requires 1/5 hours of skilled labor for each hour of unskilled labor.

It is worth noting that saying something is "intensive" in a factor is a relative statement. Saying a production process is intensive in a factor only makes sense when compared to another production process.

Unit costs

An important part of our analysis of vertical FDI will be the cost of producing a good (or a part of a good). When we had only labor as an input in our models of horizontal FDI, the unit cost was w/φ, the wage divided by the firm's productivity. With more than one input, things are slightly different.

Let w_u be the wage of unskilled labor and w_s be the wage of skilled labor. To compute the unit cost of production, we need to work out how much it costs to produce one unit of the good. From (1), we know that to produce one unit of output, we need θ_u units of unskilled labor and θ_s units of skilled labor. The cost of one unit of output, c is

$$c\left(w_u, w_s\right) = \theta_u w_u + \theta_s w_s. \tag{2}$$

Notice that we wrote the unit cost as a function of the wages, $c\left(w_u, w_s\right)$. When wages differ across countries, so will the cost of production. This will give the firm an incentive to move different parts of production to places with different relative wages.[2]

[2] In trade theory, these forces give rise to the Heckscher-Ohlin model of trade.

Numerical example, continued. Consider the production of two different goods, a and b. The production of these goods follows (1), but with different unit labor requirements: $\theta_{ua} = 5$ and $\theta_{sa} = 1$ for good a and $\theta_{ub} = 1$ and $\theta_{sb} = 10$ for good b.

1. What are the unit costs of a and b when factor prices are $w_u = 7$ (dollars per hour) and $w_s = 25$ (dollars per hour)? Call this location 1.

   The unit costs are:

$$c_a(w_u, w_s) = 5 \times 7 + 1 \times 25 = 60$$
$$c_b(w_u, w_s) = 1 \times 7 + 10 \times 25 = 257$$

2. Suppose a new production location becomes available. At this location, factor prices are $w_u = 2$ (dollars per hour) and $w_s = 30$ (dollars per hour). Call this location 2.

   What are the unit costs of a and b? Compared to the location with $w_u = 7$ and $w_s = 25$, where would you like to produce components? Where would you like to assemble?

   In the second location,

$$c_a(w_u, w_s) = 5 \times 2 + 1 \times 30 = 40$$
$$c_b(w_u, w_s) = 1 \times 2 + 10 \times 30 = 302$$

   Produce components in location 1 (257<302) and assemble them in location 2 (40<60).

Two-stage production

Now that we have a handle on Leontief production functions, let's build up the production structure of a good that is made in two stages: First, component parts (b for circuit board) are built; Second, the component parts are assembled (a for, well, assembly) into the final product. Both component parts and assembly use skilled and unskilled labor,

$$x_a = \min\left\{\frac{\ell_{au}}{\theta_{au}}, \frac{\ell_{as}}{\theta_{as}}\right\} \tag{3}$$
$$x_b = \min\left\{\frac{\ell_{bu}}{\theta_{bu}}, \frac{\ell_{bs}}{\theta_{bs}}\right\}. \tag{4}$$

The final good x is made up of one unit of components and one unit of assembly services,

$$x = \min\{x_a, x_b\}. \tag{5}$$

Notice that the final good production function has the same form as (1), but with the unit input requirements set to one. To find the cost of producing the final good, apply the cost formula, but use the unit costs of a and b,

$$c(w_u, w_s) = c_a(w_u, w_s) + c_b(w_u, w_s).$$

Numerical example, continued. Given the unit costs of a and b in locations 1 and 2, what is the unit cost of the final good when the entire good is made in location 1? Location 2?

When made entirely in location 1, the unit cost of the final good is $60 + 257 = 317$.

When made entirely in location 2, the unit cost of the final good is $302 + 40 = 342$.

When each part of the final good — assembly and components — can be carried out in the cheapest location, what is the unit cost of the final good?

When components are made in location 1 and assembled in location 2, the unit cost of the final good is $257 + 40 = 297$.