Simple Risk Sharing

[*My notes are in beta. If you see something that doesn't look right, I would greatly appreciate a heads-up.*]

In this course we have dealt primarily with static models, or models where intertemporal trade was not the first-order issue. There is a very large literature — sometimes called international finance, international business cycles, or international "macro" — in which intertemporal trade is the key mechanism. This note is a simple primer to get you thinking about some of these issues.

The big takeaway is that unbalanced trade (net export deficits or surpluses) are expressions of cross-country borrowing and lending. Why do countries borrow and lend with each other? A simple and appealing idea is that they do so to share country-specific risk.

Static risk sharing[1]

This static problem yields a lot of intuition that carries over to more complicated problems. There is one date and S states of the world, indexed by s. The probability of s occurring is $\pi(s)$. The economy is made up of countries labeled $i = 1 \dots I$. There is one good, y, and each economy receives a state-dependent endowment of the good, $y_i(s)$. Markets are complete.

The social planner's problem is to choose consumption for each country in each state of the world to maximize the weighted sum of the countries' utilities,

$$\max_{c_i(s)} \sum_{i=1}^{I} \omega_i \sum_{s \in S} \pi(s) u_i\left(c_i(s)\right) \tag{1}$$

subject to the feasibility constraint that total consumption cannot be more than total production

$$\sum_{i=1}^{I} c_i(s) \leq \sum_{i=1}^{I} y_i(s) \qquad \text{for each } s \in S. \tag{2}$$

The ω_i are the weights the planner places on each country, and $u_i(c)$ is differentiable and strictly concave. The maximization problem is straightforward. Let the Lagrange multipliers (one for each feasibility constraint) be

$$P(s) = \pi(s)\, p(s)\,, \tag{3}$$

so the Lagrangian is (substitute the constraints into the objective and sum over s),

$$L = \sum_{s} \pi(s) \sum_{i} \left(\omega_i u_i\left(c_i(s)\right) + p(s)[y_i(s) - c_i(s)]\right). \tag{4}$$

Let $u_{ic}(c)$ be the derivative of u_i with respect to c. The first-order conditions are

$$\omega_i u_{ic}\left(c_i(s)\right) = p(s) \qquad \text{for all } i, s \tag{5}$$

The left hand side is the weighted marginal utility of country i, and the right hand side is not

[1] These notes are based on notes from, and conversations with, David Backus, http://pages.stern.nyu.edu/~dbackus/. There is a lot of great material on his website. All the mistakes in this note are, obviously, mine.

country specific. The social planner equalizes the weighted marginal utilities across countries. [To see this explicitly, take the ratio of (5) for countries i and j.]

Identical power utility

Let $u_i(c) = c^{1-\alpha}/(1-\alpha)$, with $\alpha > 0$. [Refresher: The Arrow-Pratt measure of relative risk aversion is $-cu_{cc}(c)/u_c(c)$. In intertemporal choice problems this can be confounded with the intertemporal elasticity of substitution, but we are cool here.] Note that we no longer need an i on the utility function. Now (5) becomes

$$\omega_i(c_i(s))^{-\alpha} = p(s) \tag{6}$$

$$c_i(s) = \left(\frac{\omega_i}{p(s)}\right)^{1/\alpha} \tag{7}$$

Sum this over all i and use the feasibility constraint,

$$\sum_i c_i(s) = \sum_i y_i(s) = \sum_i \left(\frac{\omega_i}{p(s)}\right)^{1/\alpha} \tag{8}$$

$$p(s) = \left(\sum_i y_i(s)\right)^{-\alpha} \times \left(\sum_i \omega_i^{1/\alpha}\right)^{\alpha} \tag{9}$$

The multiplier is decreasing in the size of the aggregate endowment. This should make intuitive sense, as the multiplier is the value of relaxing the feasibility constraint. [Hint: The multiplier is "p" for a reason.] Substitute (9) into (7),

$$c_i(s) = \frac{\omega_i^{1/\alpha}}{\sum_i \omega_i^{1/\alpha}} \sum_i y_i(s)$$

$$c_i(s) = \omega_i^* \times y(s) \tag{10}$$

where $y(s)$ [no subscript] is the aggregate endowment, and ω_i^* is a normalized Pareto weight. Each country gets a share of the aggregate endowment that corresponds to this normalized weight. Notice, also, that the sharing rule is not state-dependent: The original Pareto weights were not functions of s, so neither is the sharing rule.

Even in this static problem, we can start thinking about "unbalanced trade." Let the net exports of country i in state s be $nx_i(s) = y_i(s) - c_i(s)$. This gives us

$$nx_i(s) = y_i(s) - y(s)\omega_i^* \tag{11}$$

so net exports are positive when

$$\frac{y_i(s)}{y(s)} > \omega_i^*. \tag{12}$$

When a country's share of the total endowment exceeds its "weight," it runs a trade surplus and sends goods to countries whose share of the total endowment was less than their weights (those countries are running trade deficits). That's about 98 percent of the intuition behind multi-country dynamic models.

Dynamic risk sharing

This is a slightly simplified version of Backus, Kehoe, and Kydland (1992). The intuition from the static model carries over into the dynamic model. We need to make a few changes to the environment. There are now an infinite number of periods indexed by t. In each period the set S_t events can occur. Let the *history* of events that have occurred up to period t be denoted $s^t = (s_0, s_1, \ldots, s_t)$ and the probability that we have reached this history be $\pi(s^t)$. The set of histories that can occur at time t is S^t. [If this notation confuses you, draw out the event tree. A history is one particular path (branch) from date 0 through date t.]

A two-country social planner's problem is a straight forward extension of the static problem:

$$\max_{c_1(s^t), c_2(s^t)} \omega_1 \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right) u\left(c_1\left(s^t\right)\right) + \omega_2 \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right) u\left(c_2\left(s^t\right)\right) \tag{13}$$

$$\text{s.t. } c_1\left(s^t\right) + c_2\left(s^t\right) \leq y_1\left(s^t\right) + y_2\left(s^t\right) \text{ for all } s^t \in S^t. \tag{14}$$

The dynamic version of (3) is $P(s^t) = \beta^t \pi\left(s^t\right) p\left(s^t\right)$. We can form the Lagrangian and solve this problem easily. The consumption allocations are basically the same as in the static case; here we are dealing with histories rather than the one-shot states in the static model.

Competitive equilibrium

The competitive equilibrium of this model starts to give us objects that look like things we can observe in the data. The representative agent in country i solves

$$\max_{c_i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u\left(c_1(s^t)\right) \tag{15}$$

$$\text{s.t. } c_i(s^t) + \sum_{s_{t+1}} q\left(s^t, s_{t+1}\right) a_i\left(s^t, s_{t+1}\right) \leq y_i\left(s^t\right) + a_i\left(s^t\right) \text{ for all } s^t \in S^t \tag{16}$$

$$a_i\left(s^0\right) \text{ given.} \tag{17}$$

This differs from the social planner's problem in the usual ways: A country only maximizes its own utility and it does so subject to a budget constraint, not a feasibility constraint. The feasibility constraint (market clearing in goods — we need one for assets now, too) still characterizes equilibrium, but the agent doesn't explicitly take it into account. Prices take care of informing the agent about the relative scarcity in any history.

The agent has access to a set of Arrow securities. At any history, s^t, there are events that can occur at $t+1$, s_{t+1}. There exists a security for each possible history that can be reached in the next period, $a\left(s^t, s_{t+1}\right)$, that pays one unit of consumption in that history and 0 in any other history. The price of this asset is $q\left(s^t, s_{t+1}\right)$.

The first-order conditions are

$$c_i(s^t): \; \beta^t \pi\left(s^t\right) u_c\left(c_i(s^t)\right) = \pi\left(s^t\right) p_i\left(s^t\right) \beta^t \tag{18}$$

$$a_i(s^t, s_{t+1}): \; \beta^t \pi\left(s^t\right) q\left(s^t, s_{t+1}\right) p_i\left(s^t\right) = p_i\left(s^{t+1}\right) \pi\left(s^{t+1}\right) \beta^{t+1}. \tag{19}$$

The second condition gives us the asset pricing equation

$$q\left(s^t, s_{t+1}\right) = \frac{p_i\left(s^{t+1}\right)}{p_i\left(s^t\right)} \pi\left(s^{t+1}|s^t\right)\beta. \tag{20}$$

Since the price of an asset has to be the same in each country, this condition implies that the multipliers will be the same across countries, too: The intertemporal ratio of marginal utilities will be equalized across countries. Using (18), we can replace the multipliers with marginal utilities of consumption, and have consumption asset pricing.

Writing down the competitive equilibrium problem makes explicit the balance of payments condition. The budget constraint can be rearranged to form

$$\sum_{s_{t+1}} q\left(s^t, s_{t+1}\right) a_i\left(s^t, s_{t+1}\right) - a_i(s^t) = y_i(s^t) - c_i(s^t) \text{ for all } s^t \in S^t. \tag{21}$$

The right hand side of this equation is the flow of physical goods: *net exports*. The left hand side of this equation is the matching flow of securities that makes the unbalanced trade possible (the IOUs): This is the change in country i's net foreign asset position. [With a few tweaks this is also the basis of the current account and the financial account (which are the official balance of payments aggregates), but those are details to be worked out later.]

- If trade is balanced, which in this one good world means that country i consumes exactly its endowment, then there is no change in the country's asset position.
- If the country consumes less than it was endowed, it is running a net export surplus, and the RHS is positive. The left hand side is also positive: the country has increased its claims on consumption in the future.
- Those running net export deficits are then, decreasing their net foreign asset position.

This is an important concept that is often misunderstood by the public, who tend to think of trade deficits as being universally bad. The deficit itself can't be bad: The rest of the world is sending us more stuff than we are sending them. The potential problem has to be that trade deficits are matched by promises to pay back — with interest — the extra consumption at a future date.

From here...

We've just covered the absolute basics: The idea was to work through the intuition behind these kinds of models. There is a giant literature that takes up lots of things that we didn't get into: production, multiple goods (so we can talk about the terms of trade), incomplete financial markets, imperfect information, defaultable debt, government policy, models with money, models with explicit firms and export entry costs, and much more.

Backus, David K., Patrick J. Kehoe, and Finn E. Kydland (1992). "International Real Business Cycles." *Journal of Political Economy* 100 (4), pp. 745–775.