AN EXAMINATION OF THE USE OF A SYMBOLIC SCHEMA TO TEACH STUDENTS TO SOLVE PROBLEMS INVOLVING RATES – Aaron Orzech, Tom Kaczmarek, Louis Smadbeck – June 2009

CONTENTS:

BIBLIOGRAPHY

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EXECUTIVE SUMMARY

High-school freshmen were taught to solve rate problems using a symbolic schema to represent the different values that occur in such problems. As problems increased in difficulty, students were asked to comment on the differences between easier and harder problems. Students were evaluated before and after being taught to use the schema. It was observed that while the schema did not seem to improve student understanding, it did improve the ability of weaker students to organize their work and work more accurately. It also gave some students a language that they could use to discuss a problem, and to explain why they found it difficult.

PURPOSE AND BACKGROUND

Our purposes in undertaking this project fall into two categories. First, we wanted to see what would be involved in applying academic research to our teaching practice, and specifically to find a structured way to explore the relationship between teaching for understanding and teaching for procedural fluency. More fundamentally, however, we wanted to undertake a modest piece of "action research" as a department in order to provide an occasion for collective examination and discussion of our practice, and to gain a better appreciation of what is involved in such a project in terms of time and planning, of what kinds of useful data can be gleaned from such a project, and of how to better plan such projects in the future.

We chose two articles from the NCTM's Journal for Research in Mathematics Education (see Appendix A for articles and a brief comment on our use of them.) One article explored the use of a "schema" to teach middle school students to solve problems involving ratios. The other studied students' application of the technique of cross-multiplication in inappropriate settings. We gathered questions involving rates from past New York State Integrated Algebra Regents exams, and sorted them by type and level of difficulty (see Appendix B). The most basic problems involved finding the speed of an object, given an amount of time and a distance traveled in that time, without converting units. More challenging problems required students to solve for time or distance given a speed, to work with rates other than speed, to translate phrases such as "two and a half hours" into numerical form, to convert between different units of measurement (minutes and hours), and to work with affine relations[1]. Other problems asked students to recognize graphical representations of rates as slopes.

The setting for our research was four classes of ninth grade students, all being taught by the same teacher. The classes were more or less heterogeneous in terms of level, although some skew higher or lower. The teacher described the students as having wide range of ability levels, with the lowest "unable to reliably perform basic computations with integers," while the highest were "ready to pass the Integrated Algebra Regents at the beginning of ninth grade." The initial proposal for the project was to find a way of teaching basic rate problems that would enable students to gain an understanding of rate sufficiently abstract as to allow them to recognize it in its different instantiations on the Integrated Algebra Regents (and beyond,) and to solve problems that involved rates. It was decided that this goal was too ambitious, and would be overwhelming for the students it was most intended to help. We settled on using a symbolic system, based on the schema described in Yan Ping Xin's JRME article, to help students to identify the key pieces of information in rate problems, and to organize the information into an equation which they could then solve algebraically. This method was to be applied to basic speed problems, and then extended to structurally similar problems involving other rates, rate problems involving changes of units (on which separate instruction was given), and problems involving affine relations. Before instruction took place, students were tested on basic speed/time/distance problems, and on a rate problem requiring a unit conversion – quizzes are included in Appendix C. They were also

[1] "Affine" describes a function whose graph has a constant slope and a y-intercept (or constant) that may be non-zero. A popular example of such a function is a monthly cell-phone charge composed of a fixed monthly rate, plus a per-minute charge that varies with the monthly usage. While technical, the term "affine" provides an economical way to describe such a situation, and will be used henceforth without additional explanation.

asked to describe the difference between the basic problems and the unit conversion problem. Following three days of instruction on the symbolic system and on rate conversion, students were tested again on the types of problems in the original quiz, and also on a problem involving an affine relation. They were asked to describe the difference between the affine problem and the others. Students were permitted to use a calculator on both quizzes.

The classroom teacher's lesson was observed by a department colleague for one of his four teaching periods during each of the three days of instruction. Observation write-ups and lesson plans are included in Appendix D. To briefly describe the method, students were taught to use a triangle, square and circle, respectively, to identify the numerator, denominator and speed (or other rate) in a word problem. They then used the formula "triangle over square equals circle" to organize the information into an equation and solve for the unknown quantity. The lessons went quite smoothly during the periods observed, and students seemed to be able to do the problems on which they were being instructed when called upon to do so. One interesting exchange took place when a student raised her hand and asked "is this a real formula?" The teacher asked for clarification, to which the student replied "like is it used in the real world?" The teacher explained that it was not a generally known formula, but that it would make it so easy for students to solve rate problems that they would not care. This is an amusing anecdote, but it offers a glimpse into students' awareness of and thinking about the legitimization of mathematical terminology and ideas.

GENERAL CONCLUSIONS

We concluded that the most valuable data that we could gather from this project were the individual before and after samples of student work. We found it impossible to disentangle the various effects at work in the aggregated group data (see Appendix E). There was a sharp decline in average quiz scores from the first quiz to the second one, but the classroom teacher felt strongly that this was due to factors such as the second quiz being given to some students on a Monday, and to one large class being "unmotivated" on the day of the second quiz due to extraneous factors. Upon further review, as a department, we also felt that small but significant variations in the composition of the two quizzes made the second quiz more difficult than the second one in subtle ways[2]. Our overarching conclusion about the data, however, was that it would be difficult or impossible to draw very many meaningful conclusions about the effect of the treatment from the aggregated data, and that any conclusions we did draw would have to be supported by careful examination of individual cases. Even trying to control for extraneous class-level effects, without many more data points, and a control group that was instructed by some other method, we did not think that statistical analysis was going to be a fruitful route to take. That said, while our final appraisal of the treatment was not entirely positive, we did not see reasons to conclude that the decline was caused by the treatment.

We also observed a large increase in student performance on the problem involving unit conversion, but upon further examination of the questions we concluded that this problem was significantly easier on the second quiz than on the first one. The first quiz's problem said that a song had 120 beats per minutes and asked how many beats there would be in 0.75 hours, while the second quiz's problem said that a current alternated at 120 cycles per second and asked how many cycles there would be in one minute.

The most striking phenomenon that we observed in the aggregated data was that students who performed poorly on the first quiz were much more likely to adopt the symbolic method that they were taught prior to the second quiz, and that it seemed to improve their results on the basic problems. The non-adopters had an average score of 67% on the first quiz, while the adopters had an average score of 51% on the first quiz. The non-adopters showed a decline of 20% on the basic problems, while the adopters showed an increase of 5% on the basic problems (against an overall backdrop of sharp declines). Examination of individual examples of student work bears out the conclusion that adopting

[2] Specifically, it was felt that the three basic problems on the first quiz followed a logical progression in order of difficulty, while the two basic problems on the second quiz replicated the two more difficult basic problems from the first quiz. Additionally, the quantity of "two and a half hours" was used in a multiplication problem on the first quiz, while it was the denominator of a quotient on the second quiz. Across the four classes, on the two quizzes, two-thirds of students got the correct answer on each of the five problems in question, but it was difficult to determine if this was due to better student work on a more difficult quiz, an averaging of improved performance by some students and worse performance by others, no net effect, or some combination of these. We did, however, decide that studying the exact level of difficulty of different problems would be an interesting area for further research.

the symbolic method was helpful to students who struggled with basic rate problems prior to learning the method, although exactly how it was helpful was not always clear (see Case Study #3, for example).

We did not find convincing evidence that learning the symbolic method gave students something that they could extend to help solve more challenging problems, and there is some evidence that the method may have been an obstacle to some students. There was some reason to think, however, that the method could provide a basis for reflection and group discussion about different types of rate problems, and the different methods required to solve them. This is explored in more depth in the case studies.

There was also interesting information contained in the student comments on what made the more difficult problems different from the basic ones (see Appendix E for student comments), although it was difficult to interpret and we made limited use of it in our discussion. On the first quiz, 22 students commented on essentially superficial differences, such as the scenario being described, or the presence of decimals in the more challenging problem (one may disagree as to whether these are equally superficial observations.) On the second quiz only nine students made comments to this effect. Interesting, however, slightly more than half of the students remarking on superficial differences scored between 75% and 100%, with most of the remainder scoring at least 50%. Fifteen students commented on one or both quizzes that the problems were similar because they were all rate problems. Eight students commented on one or both quizzes that the last problem was "harder." Fifteen students also made some sort of comment to the effect that a different method was required, although few of the comments offered a cogent explanation of the difference.

The comments that we found most unsettling were on the second quiz. Two students commented that the problem involving an affine relation had "three triangles." One of these comments was from student #36 who scored 100% on the first quiz and 75% on the second, while the other was from student #51 who scored 75% on the first quiz and 25% on the second. This suggested that the symbolic method prevented some students from reading the problem thoroughly and fully assimilating the information given therein.

The direct application of our findings for future instructional practice will be in Integrated Algebra classes. Next year our department is planning to offer two and four semester versions of Integrated Algebra to students of varying levels, to be taught by the same teacher. This project has led us to conclude that for the two semester students the use of this sort of symbolic method may prove a hindrance in approaching the widest possible range of rate problems, while taking time that could be better used teaching and reinforcing more advanced skills, such as unit conversion and dimensional analysis. For four semester students we are planning to teach the symbolic method as a way of establishing a baseline level of proficiency with rate problems. This can then be used to provide a basis for reflection and discussion on the limitations of the method, leading to discussion of how to approach problems that vary from the most basic form of rate problem.

On the level of our departmental practice, this project has not only reinforced the usefulness of bringing academic research to bear on our practice, and of reflecting on student work, but has helped us to think about appropriate norms for doing so. We used the two research articles here largely as a jumping off point for our own discussion. The articles helped us to frame a question that had already occurred to us in some form, namely, what is the relationship between teaching for understanding and teaching for procedural fluency? The setting for academic research projects is often quite different from that in which we operate, and we were not interested in trying to replicate experimental results *per se*. We were somewhat surprised by how many extraneous variables seemed to interfere with what we were trying to measure, and by the difficulty of determining the validity of an assessment tool (in terms of whether it is measuring what it is intended to measure). Subtle changes in how problems were phrased seemed to significantly impact the difficulty of the problems, and the very fact of these difficulties led us to question how well our students truly understood what we were trying to teach them. We are already in discussions with our school's English department about working together next year on studying and developing students' literacy skills in approaching quantitative material, but this project underlines the importance of this and provides a concrete basis for our cross-disciplinary work. A final discovery that we made in the course of this project was how rich student work can be when examined at length and in a collegial setting. Although our past practice has gone well beyond simply marking student work as right or wrong, we have not had past experience examining student work in a sustained way, in dialogue with colleagues, in the context of a specific question and in comparison with other student work. In light of this experience, we are likely to make more time do so as frequently as possible in the future.

CASE STUDY #1 – STUDENT #6

This student is characterized by her teacher as an average to below average student. One the initial quiz she did not attempt the first question, correctly solved the second question using a brute force (additive) method, and answered the third question in a way that reflected little understanding of what the question was asking. She solved the fourth problem, which was formally similar to the second problem, using a multiplicative method – her answer was incorrect only because she did not deal with the two different units used for time in the problem. This student's comment on the fourth question – "I think her problem is multiplying then the others" – is difficult to interpret. It seems to suggest that she did not notice the formal similarity between the fourth problem and the others. It also appears that the student did not detect any similarities between the first three questions.

The contrast between her initial quiz and her second quiz is striking. She successfully solved, with apparent ease, all of the problem types that carried over from the initial quiz, and her work on the affine relation problem, while ultimately incorrect, showed some correct thinking. She also successfully converted units from minutes to seconds in the third problem. This skill was covered by the teacher as a sub-part of this unit, and she used the unit-conversion method that the teacher taught the class.

Her ultimate mistake in solving the fourth problem was not subtracting the fixed monthly cost from the budget before dividing by the per-minute cost to find the number of minutes available. It appears that she gave up on her attempt to use the symbolic method to solve problem four, and her comment – "It was hard to know which one was the triangle and square" – supports this reading of her work. It appears that she may have been misled by the variable charge being written as "$0.06 per minute," looking for a printed word to give the units for the numerator in the rate. Once she abandoned this method, however, she proceeded to divide the total budget ($26) by the per-minute rate without resorting to any cross multiplication or symbolic manipulation as she did in the other problems. This would seem to reflect a level of mastery not reflected anywhere else in this student's work.

While one might argue that the symbolic method is a hindrance to this student in the affine relation problem, it would appear that this may be outweighed by the benefit to the student reflected in her performance on the first three problems. Moreover, her proficiency with the symbolic method might provide an abstract language to use in talking with her about the difference between this type of problem and the more basic ones.

CASE STUDY #2 – STUDENT #38

This student is characterized by his teacher as being one of the very strongest math students in the ninth grade. He had persistent difficulty through both quizzes dealing with time measurements, although he is clearly aware of the problem of unit conversion and is able to deal with it, although sometimes incorrectly. In the first problem on the initial quiz he divides 210 feet by 60, rather than the 3 seconds given in the problem. He then wrote "3.5 mph in feet per second" as his answer, before writing "The ball's speed in feet per second is 3.5." We were unable to make a confident guess about his reason for using 60 in the problem. In the fourth problem he correctly converted 0.75 hours to 45 minutes, but proceeded to divide the rate by 0.45, rather than (correctly) multiplying it by 45. He did not have time to finish commenting on problem four.

On the second quiz he adopted the symbolic method, and while he was able to use it proficiently it did not seem to improve his performance over the first quiz. In the second problem he made the error of writing "two and a half hours" as "2.30 hours" in his calculations – a translation that he executed correctly on the first quiz. Like student #6, he struggled to apply the symbolic method on the fourth problem, and eventually seemed to give up, simply dividing $26 by $0.06 to arrive (incorrectly) at 433.33 minutes.

We believe that this student would have been better served by instruction on dealing with units, particularly time units. Moreover, he is clearly ready for such instruction, as he seems to be aware of the problem and struggling to work it out on his own. As in the case of student #6, there is some cause to think that attempting to apply the symbolic method hindered a more thorough reading and consideration of the problem at hand. In this case, however, there is more reason to think that the (hypothetical) harm done by the symbolic method might actually outweigh its benefits.

CASE STUDY #3 – STUDENT #57

This student is described by her teacher as "very low functioning." Her work on the first quiz is almost entirely wrong, and does not reflect a clear grasp of what is being asked. Her answer to problem four does use multiplication correctly, but neglects to deal with the two different time units used in the problem. She does not detect the formal similarity to problem two, and we were unable to make any meaningful guess as to how she arrived at her answer to problem two.

On the second quiz, she appears to faithfully adopt the symbolic method, and to fluently obtains correct answers to the two basic problems at the beginning of the quiz. Upon closer inspection, however, her application of the method to deciphering the word problem appears haphazard, and ultimately incorrect. She assigns the shapes incorrectly to the different pieces of information, and never circles the part of the question that designates the rate. Nonetheless, in the first problem she proceeds to arrange the information into a correct equation, which she solves correctly by cross multiplication. In the second problem she sets up the equation incorrectly, and seems to arrive at the correct answer essentially by accident. In the third problem on the second quiz her application of the method is again incorrect, and the equation that she uses is entirely incorrect – no acknowledgement is made of the two different time units being used. She makes no attempt to answer the fourth problem on the second quiz, and states that the problem is different because it "does not have … per minute," which it in fact does.

This student seemed to find it helpful to have a schema for organizing her work (although it may also be helping her as a literacy tool). She shows clear improvement over the first quiz, and her work on the second quiz would be likely to provide a more helpful basis for discussing her misunderstandings with her, and remediating those misunderstandings. There is also some reason to think that the instruction that she received in working with units is helping her to organize information when her application of the symbolic method is shaky, although it would be easier to determine whether this is the case by having her explain verbally how she is working through the problems.

CASE STUDY #4 – STUDENT #26

This student, like #57, is described by her teacher as "very low functioning." On the first quiz she answered the first problem correctly, but answered the second problem, which asks for a distance, with "3600 miles per hour" and the third problem, which asks for an amount of time, with "45 miles." The numbers are also incorrect. Her error in the second problem was to decompose the time (two and a half hours) into 2 and 30 and multiply by these two numbers in sequence. She is correct in multiplying the speed by two, but she then proceeds to multiply the result (120) by 30. This seems to reflect an almost complete lack of understanding of the procedures that she applies. Her work on problems three and four, likewise, show little or no understanding of the procedures that she attempts to apply.

On the second quiz, she applies the symbolic method, and solves the first problem (which corresponds to the second problem on the first quiz) correctly. Her work on the second problem is also formally correct, although she inexplicably writes "two and a half hours" as 2.125. On the third problem she stops short, appearing to recognize the issue of two different time units being used in the problem. She does not attempt the fourth problem.

We had some debate as to whether this student should be considered an adopter, although she does use the symbolic method in the first problem and in part of the third. One point of interest here is that while the student's score (25%) is the same on both quizzes, and would lead one to believe that she had learned nothing, even a brief examination of the second quiz reveals it to be within striking distance of 75%, while one would be hard pressed to give any sort of partial credit above the baseline 25% on the first quiz.