Bott Periodicity

Literacy Winter 2014

I. Intro

- $U_n = \{A \in GL_n\mathbb{C} \mid AA^* = I\}$ $\quad SU_n = U_n \cap SL_n\mathbb{C}$
- $U_n \hookrightarrow U_{n+1}$ $\quad U := \bigcup_{n=1}^{\infty} U_n$

  $A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$

Basic Problem: Compute $\pi_i(U_n)$ (hard)

- Fix i.

$$\pi_i(U_1) \to \pi_i(U_2) \to \cdots \to \pi_i(U_n) \to \pi_i(U_{n+1}) \to \cdots$$

eventually isomorphisms. Stable group is $\pi_i(U)$.

Theorem (Bott Periodicity) $\pi_i(U) = \begin{cases} 0 & i \equiv 0 \bmod 2 \\ \mathbb{Z} & i \equiv 1 \bmod 2. \end{cases}$

Exercise: Prove for $i = 1,2,3$ (use LES for $U_{n-1} \to U_n \to S^{2n-1}$ and $U_1 \simeq S^1$ $SU_2 \simeq S^3$)

30 sec Pf of BP: Fix i and take $n = 10i$. By stability suffices to show $\pi_i(U_n) \simeq \pi_{i+2}(U_n)$. Consider

$\Omega = \Omega_{I,I}(SU_{2n})$ based pw smooth loops

$\Omega_{I,-I} = \Omega_{I,-I}(SU_{2n})$ pw smooth paths from I to $-I$.

$\Omega^{min} \subset \Omega_{I,-I}$ global minimizers for length/energy

$S^3 \simeq SU_2$  $\Omega^{min} \simeq S^2$.

We'll show

$$\pi_i(U_n) \cong \pi_{i+1}(Gr_n \mathbb{C}^{2n}) \cong \pi_{i+1}(\Omega^{min}) \cong \pi_{i+1}(\Omega_{I,-I}) \cong \pi_{i+2}(SU_{2n})$$
$$\cong \pi_{i+2}(U_{2n})$$
$$\cong \pi_{i+2}(U_n).$$

Purple iso's are by LES of fibration

Green iso's are by Morse theory.

II. SU_{2n} as a Riemannian mfld.

- Lie algebra: $T_I SU_{2n} \cong su_{2n} \subset M_{2n}\mathbb{C}$ traceless anti-Hermitian ($x^*+x=0$) matrices

Frobenius inner product for $x, y \in su_{2n}$ $\langle x,y\rangle = tr(xy^*)$.

For $g \in SU_{2n}$ $\langle gxg^{-1}, gyg^{-1}\rangle = \langle x,y\rangle$.

- Riemannian metric: left translation of $\langle\, ,\rangle$ defines bi-invariant R. metric on SU_{2n}.

- Geodesics: Riem exponential = Matrix exponential

$\Rightarrow$ geodesics = 1-parameter subgroups.

III. The iso $\pi_i(Gr_n \mathbb{C}^{2n}) \cong$ ~~[illegible]~~. $\pi_i(\Omega^{min})$.

<u>Theorem</u> ~~Ω^{min} is homeo~~ $\Omega^{min} \cong Gr_n \mathbb{C}^{2n}$ homeomorphic

<u>Proof for SU_2</u> (not using $SU_2 \cong S^3$!)

- <u>Step 1</u> What are geodesics from I to $-I$?

Let $x \in \mathfrak{su}_2$, $\gamma : [0,1] \longrightarrow SU_2$, $t \longmapsto \exp(tx)$. ~~with $\gamma(1) = I$~~

with $-I = \gamma(1) = \exp(x)$.

- For ~~case~~ x diagonal $x = \begin{pmatrix} ia & \\ & -ia \end{pmatrix}$ $\exp(x) = \begin{pmatrix} e^{ia} & \\ & e^{-ia} \end{pmatrix} = -I$

$\Rightarrow$ a odd multiple of π.

- For ~~case~~ general x, x anti-hermitian $\Rightarrow$ x normal $\Rightarrow$ x diag'ble
  $xx^* = x^*x$

Let $g \in SU_{2n}$ st. gxg^{-1} diagonal.

$$\exp(gxg^{-1}) = g\exp(x)g^{-1} = g(-I)g^{-1} = -I$$

$\Rightarrow$ x has evals of form $ik\pi$ k odd

So $\left\{ \begin{matrix}\text{Geodesics} \\ I \text{ to } -I\end{matrix} \right\} \cong \{ x \in \mathfrak{su}_2 \mid \text{evals are of form } ik\pi,\ k \text{ odd} \}$.

- Step 2 Minimal geodesics

Energy $E(\gamma) = \int_0^1 \|\dot\gamma(t)\|^2 \, dt = \|\dot\gamma(0)\|^2$

$$= \langle x, x \rangle$$

$$= tr(xx^*)$$

$$= tr\begin{pmatrix} ik\pi & \\ & -ik\pi \end{pmatrix}\begin{pmatrix} -ik\pi & \\ & ik\pi \end{pmatrix} = tr\begin{pmatrix} k^2\pi^2 & \\ & k^2\pi^2 \end{pmatrix} = 2k^2\pi^2$$

Energy minimized when $k = \pm 1$.

$$\Omega^{min} = \{ x \in \mathfrak{su}_2 \mid \text{evals of } x \text{ are } \pm i\pi \} =: X^{\pm\pi}$$

SU_{2n} acts on $X^{\pm\pi}$.

All matrices of $X^{\pm\pi}$ conjugate to $x_0 = \begin{pmatrix} \pi & \\ & -\pi \end{pmatrix}$ so SU_2 acts transitively.

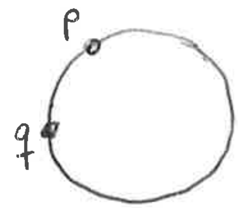
Stabilizer of $x_0 = S(U_1 \times U_1) = \left\{ \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} : A_i \in U_1,\ \det A_1 \det A_2 = 1 \right\}$

Then $\Omega^{min} \cong \dfrac{SU_2}{S(U_1 \times U_1)} \cong Gr_1 \mathbb{C}^2$.

IV. The iso $\pi_i(\Omega^{min}) \cong \pi_i(\Omega_{I,-I})$.

Two approaches to topology of $\Omega_{p,q}(M)$.

1) Choose p,q nonconjugate



$E : \Omega_{p,q} \longrightarrow \mathbb{R}$

Hess_E nondegenerate

$$\Omega_{p,q} = \bigcup_{\substack{\gamma \text{ geod} \\ \text{index } \lambda(\gamma)}} e^{\lambda(\gamma)}$$

2) Choose p,q conjugate

Thm Let M Riem mfld $p,q \in M$ conjugate. Assume Ω^{min} is a manifold. If every nonminimal geod from p to q has index $\geq \lambda$, then

$$\pi_i(\Omega_{p,q}) \cong \pi_i(\Omega^{min})$$

for $i \leq \lambda - 2$.

Recall The index of γ is the number of conjugate points along γ

$\therefore$ To ~~compute index of~~ show $\pi_i(\Omega^{min}) \cong \pi_i(\Omega_{I,-I})$ (small i) need to compute index of nonminimal geodesics.

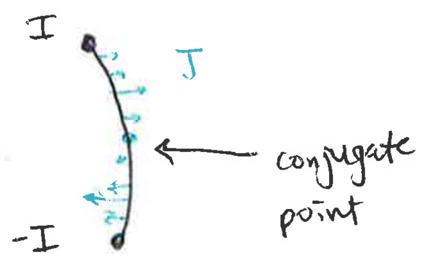
<u>Thm</u> Every nonminimal geodesic $\gamma \in \Omega_{I,-I}(SU_{2n})$ has index $\geq 2n+2$.

<u>Proof for SU_2</u> (not using $SU_2 \simeq S^3$!)

$$\mathfrak{su}_2 = \mathbb{R}\Big\{ \underbrace{\begin{pmatrix} i & \\ & -i \end{pmatrix}}_{h}, \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} & i \\ i & \end{pmatrix}}_{u_2} \Big\}$$

$[h, u_1] = 2u_2$
$[h, u_2] = -2u_1$
$[u_1, u_2] = 2h.$

Let $x = k\pi h = \begin{pmatrix} ik\pi & \\ & -ik\pi \end{pmatrix}$ $\quad \exp x = -I.$ $\quad \gamma(t) = \exp(tx)$



~~Looking for~~ Find conjugate points by finding zeros of Jacobi fields.

Let $X = X(t) = \dot\gamma(t)$. J (vf along γ) is a Jacobi field if

$$\nabla_X^2 J + R(X,J)X = 0.$$

Here $R(X(t), \cdot)X(t) : T_{\gamma(t)} SU_2 \longrightarrow T_{\gamma(t)} SU_2.$ linear

For $t=0$ $\quad R(x,\cdot)x : \mathfrak{su}_2 \longrightarrow \mathfrak{su}_2$ given explicitly by

$$v \longmapsto -\tfrac{1}{4}[x,[x,v]] = -\tfrac{1}{4}\,\mathrm{ad}(x)^2(v).$$

~~By~~

For $x = \begin{pmatrix} ik\pi & \\ & -ik\pi \end{pmatrix}$ $\quad \mathrm{ad}\, x = \begin{array}{c} \begin{matrix} h & u_1 & u_2 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2k\pi \\ 0 & 2k\pi & 0 \end{pmatrix} \end{array}$

$$-\tfrac{1}{4}\,\mathrm{ad}(x)^2 = -\tfrac{1}{4}\begin{pmatrix} 0 & & \\ & -4k^2\pi^2 & \\ & & -4k^2\pi^2 \end{pmatrix} = \begin{pmatrix} 0 & & \\ & k^2\pi^2 & \\ & & k^2\pi^2 \end{pmatrix}$$

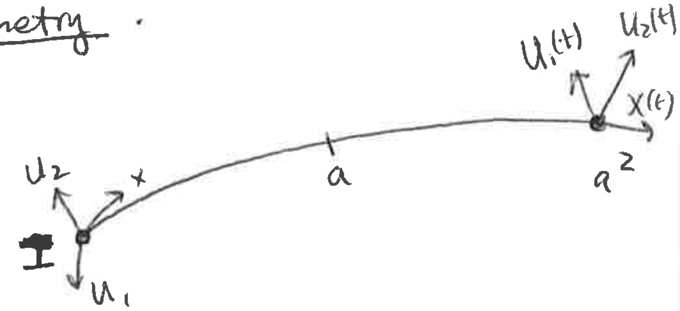
$\Rightarrow u_1, u_2$ evectors of $R(x,\cdot)x$ w/ eval $k^2\pi^2$.

Parallel transport u_i along γ to get v.f.'s U_i $i=1,2$.

<u>Claim</u> Fix $0 < t < 1$. $U_1(t), U_2(t)$ are evectors of $R(X(t), \cdot)\, X(t)$ w/ eval $k^2\pi^2$.

<u>Proof</u> Define $\sigma : G \to G$, $g \mapsto g^{-1}$, $a = \exp\left(\frac{t}{2} x\right)$, $\phi : G \to G$, $g \mapsto a g^{-1} a$.

$\phi(a) = a$, $\phi(a^2) = e$, ϕ is an <u>isometry</u>.
ϕ preserves γ



Use ϕ to translate the question to $T_I SU_2 = \mathfrak{su}_2$.
(This works b/c SU_2 a symmetric space) ✓

Write $J(t) = a_1(t)\, U_1(t) + a_2(t)\, U_2(t)$

$$0 = \nabla_X^2 J + R(X, J)X = \frac{d^2 a_1}{dt^2}(t)\, U_1(t) + k^2\pi^2 a_1(t)\, U_1(t) + \frac{d^2 a_2}{dt^2}(t)\, U_2(t) + k^2\pi^2 a_2(t)\, U_2(t).$$

so get
$$\begin{cases} a_1'' + k^2\pi^2 a_1 = 0 \\ a_2'' + k^2\pi^2 a_2 = 0 \end{cases}$$
with solutions $a_i = c_i \sin(k\pi t)$

$a_i(s) = 0$ for $s = \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}$, so $\gamma(s)$ conjugate point of multiplicity 2. If γ nonminimal $k \geq 3$. For $k=3$ get $s = \frac{1}{3}, \frac{2}{3}$ $\Rightarrow$ γ has ≥ 2 conj pts w/ mult 2
$\Rightarrow$ $\operatorname{index}(\gamma) \geq 2 \cdot 2 = 4$ $\blacksquare$