Degree Theorem

Rigidity Seminar
Summer 2014

Defn N mfld. Normalized entropy

$$\mathrm{ent}(N,\cdot) : \mathrm{RiemMet}(N) \longrightarrow [0,\infty).$$
$$g \longmapsto h(g)^n \, \mathrm{Vol}(N,g).$$

Rmk descends to $\mathrm{RiemMet}(N)/\text{scaling}$.

Thm 1 (BCG). $(M,g_0) = (\mathbb{H}^n/\Gamma, g_{hyp})$ closed

(N,g) closed

$\forall f : N \to M$ with $|\deg f| \geq 1$,

$$|\deg f| \leq \frac{\mathrm{ent}(N,g)}{\mathrm{ent}(M,g_0)} = \left(\frac{h(g)}{h(g_0)}\right)^n \frac{\mathrm{Vol}(N,g)}{\mathrm{Vol}(M,g_0)}.$$

if $\dim M \geq 3$ equality $\iff$ $f \sim$ isometric covering.

More generally M neg curved loc sym. Implies loc sym metric uniquely minimized entropy.

Entropy rigidity Conjecture: should hold $\forall\, M = \Gamma\backslash G/K$ noncpt type

Thm 2 (Connell-Farb)

$(M,g_0) = (\mathbb{H}^n \times \mathbb{H}^n/\Gamma, g_{hyp} \times g_{hyp})$ $n \neq 2$ closed

or $(M,g_0) = (SO(n)\backslash SL_n\mathbb{R}/\Gamma, g_0)$ $n \geq 4$ closed.

(N,g) closed.

$\exists C = C(\dim M)$ so that $\forall f : N \to M$ with $|\deg f| \geq 1$

$$|\deg f| \leq C \cdot \left(\frac{h(g)}{h(g_0)}\right)^n \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(M)}.$$

"More generally M loc sym sp noncpt type"

Rmks ① $C > 1$ so doesn't imply entropy rigidity conjecture.

② No statement about equality

③ $h(g)$, $h(g_0)$ depend only on Ricci curvatures of N, M

(bounding $Ric(N)$ from below bounds $h(g)$ from above)

Applications

① **MinVol**

$$\mathrm{Vol}(N, \cdot) : \mathrm{RiemMet}(N) \longrightarrow [0, \infty)$$

$$g \longmapsto \mathrm{Vol}(N, g).$$

Defn $$\mathrm{MinVol}(N) = \min \{ \mathrm{Vol}(N, g) \mid |K(g)| < 1 \}.$$

- (Gromov) Proof that MinVol is an interesting invariant.

$$4\pi + 0.01 \le \mathrm{MinVol}(\mathbb{R}^2) \le (2 + 2\sqrt{2})\pi$$

- (BCG) $\mathrm{MinVol}(\mathbb{H}^n/\Gamma) = \mathrm{Vol}(\mathbb{H}^n/\Gamma, g_{hyp})$.
- (Connell-Farb) For M in Thm 2, $\mathrm{MinVol}(M) > 0$.

Pf: Apply Thm 2 w/ $f = id$.

$$\mathrm{Vol}(M, g) \ge \mathrm{Vol}(M, g_0) \cdot \frac{h(g_0)^n}{h(g)^n} \cdot \frac{1}{C}.$$

When $|K(g)| < 1$ $h(g) < A$ so

$$\mathrm{Vol}(M, g) \ge \frac{\mathrm{ent}(M, g_0)}{AC} > 0.$$

② Simplicial Volume

(Connell-Farb + Lafont-Schmidt) $\quad M$ as in Thm 2 $\Rightarrow \|M\| > 0$.

Wouter will explain next time.

③ Self-maps

M as in Thm 2 $\Rightarrow \nexists\ f: M \to M$ with $|\deg f| > 1$.

<u>Pf</u> Take $g = g_0$ $\quad f = id$.

$\deg f \leq C$. $\quad$ but also for $n > 0$

$(\deg f)^n = \deg(f^n) \leq C$. $\quad$ impossible.

<u>Strategy of Proof</u> of Thm 2

① Lift

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

② For $s > h(g)$ define $\quad F_s: Y \xrightarrow{PS(s)} Meas(Y_\infty) \xrightarrow{f_*} Meas(X) \xrightarrow{\circledast \mu_{\dots}} Meas(X_\infty) \xrightarrow{Bar} X$

③ $|Jac F| \leq C \cdot \left(\frac{s}{h(g_0)}\right)^n$

④ $|\deg f| \cdot Vol(M) = \left|\int_N f^* dvol_M\right| = \left|\int_N F^* dvol_M\right| = \left|\int_N Jac F \, dvol_N\right|$

$\leq \int |Jac F| \, dvol_N \leq C\left(\frac{s}{h(g_0)}\right)^n Vol(N)$.

If Y not npc Y_∞ makes no sense.

Convolution of measures:

$\circledast \mu_x$ "convolution with PS measures" $\{\mu_x\}$ $x \in X$.

For $\lambda \in \mathrm{Meas}(X)$ $U \subset X_\infty$,

$$\lambda \circledast \mu_x \, (U) = \int_X \mu_x(U) \, d\lambda(x)$$

Two differences in higher rank

① PS measures have support on Furstenberg ∂. $G/P \subset \partial X_\infty$.

$SL_n\mathbb{R}/P$ $P = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ 0 & & \ddots \; * \end{pmatrix} \right\}$.

This ∂ is algebraic object and this important for Jacobian bound.

② Need new argument to define $\mathrm{Bar}: M(X_\infty) \to X$.

In neg. curvature $d(p, \cdot)$ strictly convex

$\Rightarrow B_\theta(o,x)$ strictly convex

$\Rightarrow \mathrm{Bar}(x) = \int_{\partial X} B_\theta(o,x) \, d\lambda(\theta)$ strictly convex.

But in NPC $B_\theta(o,x)$ not nec. strictly convex.

③ Jac bound more delicate.

$X = G/K = SL_3\mathbb{R}/SO_3.$

$p = eK \in X$ ~~T_pX~~

$\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p}$

$\mathfrak{g}$ = traceless 3×3

$\mathfrak{k}$ = antisym

$\mathfrak{p}$ = sym $\simeq T_pX.$

$\exists\, \mathfrak{a} \subset \mathfrak{g}$ max abelian

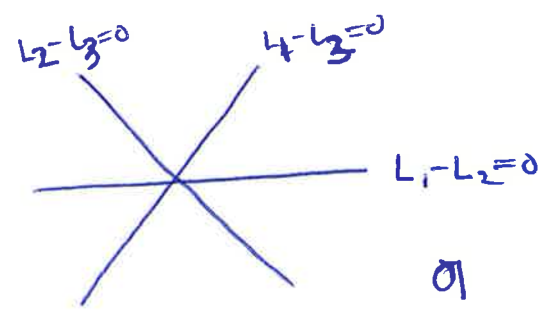
$$\mathfrak{a} = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \subset \mathfrak{p}.$$

$F = \exp(\mathfrak{a}) \subset X$ euclidean (a flat)

~~$\mathfrak{a}$ $\mathfrak{g}$ by conjugation~~ adjoint

$L_i : \mathfrak{a} \longrightarrow \mathbb{R}$

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \longmapsto a_i$$



$v \in \mathfrak{a}$ is <u>regular</u> is $(L_i - L_j)(v) \neq 0 \quad \forall i,j$

$\Leftrightarrow \quad v = \begin{pmatrix} v_1 & & \\ & v_2 & \\ & & v_3 \end{pmatrix}$ w/ $v_i \neq v_j$

$\Leftrightarrow \quad \exp(tv)$ contained in $!$ flat.

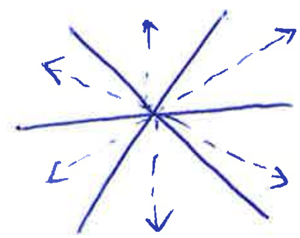
<u>Defn</u> The Furstenberg ∂ $\partial_F X$ is the G orbit of v (regular)

~~a regular vector v.~~

stabilizer of v = $P = \begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix} = AN$

$\Rightarrow \partial_F X \simeq G/P \simeq K.$

<u>Rmk</u> $\partial_F X \cap \partial F$ is finite :



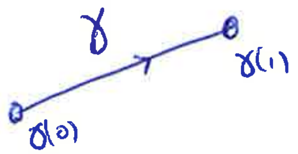
Reality check.

$\partial X \simeq S^4$ $\qquad \partial_F X \simeq SO_3 \quad \partial(F) \simeq S^1.$

Sketch proof that Bar is strictly convex. Let $\mu \in Meas(\partial_F X)$.

$$Bar(x) = \int_{\partial_F X} B_\theta(x)\, d\mu(\theta).$$

Enemy to strict convexity is ~~$B_\theta(x)$ constant~~

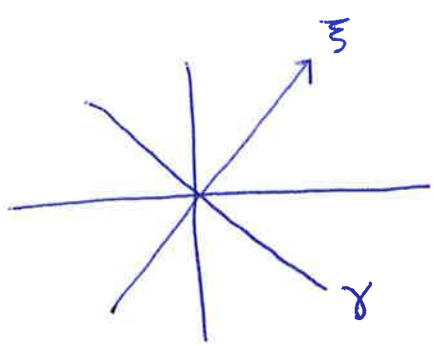


in X so that $B_\theta \circ \gamma$ constant for a.e. θ wrt μ.

$\Rightarrow$ suffices to find $\xi \in \partial_F X$ so that $B_\xi \circ \gamma$ strictly convex.

$B_\xi \circ \gamma$ constant happens in a flat

E^2



In this scenario γ is regular

$\Rightarrow \gamma$ in unique flat.

$\Rightarrow \exists$ many $\xi \in \partial_F X$ s.t. $B_\xi(\gamma)$ not constant