Gelfand-Fuks coho
Satellite diffeo seminar
11-11-14. 1

I. Review (Char classes of foliations)

<u>Setup</u>
- M mfld
- $\mathcal{F}$ foliation codim q
- $0 \to T\mathcal{F} \longrightarrow TM \longrightarrow Q \longrightarrow 0$
- ∇ connection on Q, curvature $\Omega \in \Omega^2(M; \mathfrak{gl}_n\mathbb{R})$

<u>Chern-Weil</u>

$$\mathbb{R}[c_1, \dots, c_q] \xrightarrow{\phi(\mathcal{F})} \Omega^\bullet(M)$$
$$c_i \longmapsto c_i(\nabla) = tr(\Omega^i)$$

<u>Defn</u> $\underline{\mathbb{R}}[c_1, \dots, c_q] = \mathbb{R}[c_1, \dots, c_q] / \deg > 2q$.

<u>Bott Vanishing</u> If ∇^B basic cnxn on Q and $p \in \mathbb{R}[c_1, \dots, c_q]$ $\deg(p) > 2q$, then $p(\nabla) = 0$.

- Also $c_{2i+1}(\nabla^B) = dh_{2i+1}(\nabla^B)$ since for ∇^R Riemannian cnxn

$$[c_{2i+1}(\nabla^B)] = [c_{2i+1}(\nabla^R)] = 0.$$

<u>Defn</u> $WO_q = \underline{\mathbb{R}}[c_1, \dots, c_q] \otimes \Lambda(h_1, h_3, \dots, h_{<q})$ DGA

$|c_i| = 2i$ $|h_i| = 2i-1$.

w/ $dc_i = 0$ $dh_i = c_i$

- $\mathcal{F}$ on M defines

$$WO_q \xrightarrow{\phi(\mathcal{F})} \Omega^\bullet(M)$$
$$c_i \longmapsto c_i(\nabla^B)$$
$$h_i \longmapsto h_i(\nabla^B)$$

on $H^\bullet(\ )$ get cc's of foliations

e.g. $q=1$ $WO_1 = \mathbb{R}[c_1] \otimes \Lambda(h_1)$ $H^\bullet(WO_1) = \mathbb{R} \oplus \mathbb{R}\{h_1 c_1\}$.

$$\phi(\mathcal{F})(h_1 c_1) = GV(\mathcal{F}) \quad \text{Godbillon-Vey.}$$

<u>Goal</u> Get more cc's of foliations & new perspective on above story.

II. Gelfand-Fuks cohomology

- M^n mfld, $\mathfrak{X}M$ = vf's on M, $C^\infty(M)$ smooth functions

A. de Rham cohomology

<u>Defn</u> $\Omega^k(M) = \{\mathfrak{X}M \times \cdots \times \mathfrak{X}M \longrightarrow C^\infty(M)$ $C^\infty(M)$-multilinear, alternating $\}$

with differential $d : \Omega^k \longrightarrow \Omega^{k+1}$

$$d\omega(X_0, \dots, X_k) = \sum_i (-1)^{i-1} X_i\left[\omega(X_0, \dots, \hat{X}_i, \dots, X_k)\right]$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k).$$

<u>Remarkable facts</u>

1. d preserves $C^\infty(M)$-linearity (uniquely defines d)
2. $H^i(\Omega M, d)$ finite dim'l $\forall i$. $(H(\Omega^\bullet M, d) \simeq H^\bullet(M))$

B. Lie algebra cohomology.

<u>Defn</u> $A^k(M) = \{\mathfrak{X}M \times \cdots \times \mathfrak{X}M \longrightarrow \mathbb{R}$ multilinear, alternating, cts wrt C^∞ top$\}$.

w/ differential $d : A^k \longrightarrow A^{k+1}$

$$d\omega(X_0, \dots, X_k) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots \hat{X}_i \dots \hat{X}_j \dots X_k)$$

$$H(A^\bullet M, d) =: H_c^\bullet(\mathfrak{X}M)$$ <u>Gelfand-Fuks coho</u>

<u>Rmk</u> For any Lie alg $A^k(\mathfrak{g})$ makes sense.
For $\mathfrak{g} = \mathfrak{X}M$ w/o topology, $H^\bullet(A^\bullet \mathfrak{g})$ intractable

- For G cpt lie $\quad \Omega^\bullet(G)^G \subset \Omega(G)$ subcplx

  & $(\Omega^\bullet(G)^G, d) \simeq (A^\bullet \mathfrak{g}, d)$

$$\Rightarrow \quad H^\bullet_{dR}(G) = H^\bullet(\Omega^\bullet(G)^G, d) \simeq H^\bullet(A\mathfrak{g}, d) = H^\bullet_{(c)}(\mathfrak{g})$$

__Remarkable fact__

__Main Thm__ (GF) $\quad H^i_c(\mathfrak{X}M)$ finite dim'l $\forall i$.

For $M = S^1$ $\quad H^\bullet_c(\mathfrak{X}M) \simeq \mathbb{R}[e] \otimes \Lambda(\omega)$.

$|e| = 2$ Euler class

$|\omega| = 3$ Godbillon-Vey

Explicit cocycles: $\quad f\frac{\partial}{\partial x}, \, g\frac{\partial}{\partial x}, \, h\frac{\partial}{\partial x} \in \mathfrak{X}S^1$

$$e(f,g) = \int_0^1 \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} dx$$

$$\omega(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}_0$$

For $\alpha, \beta, \gamma \in \mathrm{Diff}(S^1)$, act on $\frac{\partial}{\partial x}$ to get group cocycles $H^\bullet(B(\mathrm{Diff}\, S^1)^\delta; \mathbb{R})$.

III. GF's theorem

<u>Defn</u> $\mathfrak{a}_n = \mathbb{R}[[x_1, \dots, x_n]] \otimes \mathbb{R}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ Lie algebra of formal vector fields.

$$X = \sum p_i \frac{\partial}{\partial x_i} \qquad p_i \in \mathbb{R}[[x_1, \dots, x_n]].$$

<u>Defn</u> Define space Y_n by

$$\begin{tikzcd} U_n \arrow[d] & U_n \arrow[d] \\ Y_n \arrow[r] \arrow[d] & EU_n \arrow[d] \\ (BU_n)^{2n} \arrow[r] & BU_n \end{tikzcd}$$

For any M^n, $\begin{matrix} FM \\ \downarrow \\ M \end{matrix}$ on Framebundle.

$$Y_n \to Y_n M = \frac{FM \times Y_n}{O(n)} \xrightarrow{\pi} M$$

<u>Proof Outline</u>

<u>Step 1</u> localize $\mathfrak{X}_p M \subset \mathfrak{X} M$ vf's that vanish at p to all orders

$$\left(X = \sum f^i \frac{\partial}{\partial x_i} \quad f^i \in C^\infty(U), \quad \frac{\partial^\alpha}{\partial x^\alpha} f^i(p) = 0 \quad \forall i, \forall \alpha \right)$$

- $\mathfrak{X} M \longrightarrow \mathfrak{X} M / \mathfrak{X}_p M \simeq \mathfrak{a}_n.$
- $A_p^\bullet M := A^\bullet(\mathfrak{X} M / \mathfrak{X}_p M) \longrightarrow A^\bullet(\mathfrak{X} M) = A^\bullet M.$
- $H(A_p^\bullet M) \simeq H_c^\bullet(\mathfrak{a}_n)$
- <u>Thm</u> $H_c^i(\mathfrak{a}_n) \simeq H^i(Y_n)$ tractable, finite dim'l $\forall i$.

<u>Step 2</u> globalize/sseq $\Gamma(Y_n M)$ = sections of π.

<u>Thm</u> (Bott-Segal, Haefliger) $H_c^\bullet(\mathfrak{X} M) \simeq H^\bullet(\Gamma(Y_n M))$

Coho function spaces is tractable (Sseq)

Examples

① $M = \mathbb{R}^n$

$$\mathbb{R}^n \sim * \implies Y_n\mathbb{R}^n \simeq \mathbb{R}^n \times Y_n \implies \Gamma(Y_n\mathbb{R}^n) \simeq \mathrm{Map}(\mathbb{R}^n, Y_n).$$

$$\mathrm{Map}((\mathbb{R}^n,0),(Y_n,y)) \longrightarrow \mathrm{Map}(\mathbb{R}^n, Y_n) \xrightarrow{ev_0} Y_n \qquad \text{fix } y \in Y_n.$$

Claim ↑ contractible

Pf: $\quad f_t : (D^n, 0) \xrightarrow{1-t} (D^n, 0) \xrightarrow{f} (Y_n, y)$

$$f_0 = f \qquad f_1 \equiv y. \qquad \checkmark$$

$$\implies \Gamma(Y_n\mathbb{R}^n) \simeq \mathrm{Map}(\mathbb{R}^n, Y_n) \sim Y_n.$$

$$\implies H^\bullet_c(\mathcal{X}\mathbb{R}^n) = H^\bullet(\Gamma Y_n\mathbb{R}^n) \simeq H^\bullet(Y_n) \simeq H^\bullet(\mathfrak{a}_n).$$

② $M = S^1$

Compute Y_1

$$\begin{tikzcd} U_1 \arrow[d] & U_1 \arrow[d] \\ S^3 \arrow[r] \arrow[d] & S^\infty \arrow[d] \\ (BU_1)^{(2n)} = \mathbb{C}P^1 \, \cancel{(BU_1)^{(2n)}} \arrow[r] & \mathbb{C}P^\infty = BU_1 \end{tikzcd} \qquad \implies Y_1 = S^3$$

$$Y_1(S^1) = \frac{F(S^1) \times Y_1}{O(1)} = \frac{S^1 \times \mathbb{Z}/2 \times S^3}{\mathbb{Z}/2} = \frac{\mathbb{Z}/2 \times (S^1 \times S^3)}{\mathbb{Z}/2} \simeq S^1 \times S^3.$$

$$\implies \Gamma(Y_1 S^1) = \mathrm{Map}(S^1, S^3).$$

Compute $H^\bullet(\mathrm{Map}(S^1, S^3))$

$$\Omega S^3 \to \mathrm{Map}(S^1, S^3) \to S^3$$

$$\Omega S^3 \longrightarrow \underset{*}{\overset{}{PS^3}} \longrightarrow S^3 \qquad (PS^3 \simeq *)$$

$$H^\bullet_c(\mathcal{X}S^1) \simeq H^\bullet(\mathrm{Map}(S^1, S^3)) \simeq H^\bullet(\Omega S^3) \otimes H^\bullet(S^3)$$
$$\simeq \mathbb{R}[e] \otimes \Lambda(\omega).$$

IV. The local complex

$$A_p = A_p M = A\left(\mathfrak{X}M / \mathfrak{X}_p M\right). \qquad \mathfrak{a}_n \simeq \mathfrak{X}M / \mathfrak{X}_p M.$$

<u>Thm</u> $H^\bullet_c(\mathfrak{a}_n) \simeq H^\bullet(Y_n) \simeq H^\bullet(\text{wedge of spheres})$

<u>Thm</u> $H^\bullet_c(\mathfrak{a}_n) \simeq H\left[\!\left[\underbrace{\mathbb{R}[c_1, \ldots, c_n] \otimes \Lambda(h_1, h_2, \ldots, h_n)}_{WU_n}, d\right]\!\right]$ $\quad dh_i = c_i \quad dc_i = 0.$

<u>Thm</u> $H^\bullet_c(\mathfrak{a}_n, O(n)) \simeq H^\bullet(WO_n)$

$$\begin{tikzcd}
H^\bullet(WO_n) \arrow[r, "\phi(F)"] \arrow[d, phantom, sloped, "\simeq"] & H^\bullet(M) \arrow[d, phantom, sloped, "\simeq"] \\
H^\bullet(\mathfrak{a}_n, O(n)) \arrow[r] & H^\bullet(J_\infty M / O(n)) \arrow[d, phantom, sloped, "\simeq"] \\
 & M
\end{tikzcd}$$