Goldman's Thm   Rigidity Seminar 2014 Summer

Goals

① Understand $PSL_2\mathbb{R}$ and $\widetilde{SL_2\mathbb{R}}$

- conjugacy classes
- decomposition into hyperbolic, parabolic, elliptic

② Understand trace coordinate

- real points of $\mathrm{Hom}(F_2, SL_2\mathbb{C})/SL_2\mathbb{C} \cong \mathbb{C}^3$
- understand level sets of

$$\begin{aligned} \mathbb{R}^3 &\xrightarrow{\kappa} \mathbb{R} \\ (x,y,z) &\longmapsto x^2+y^2+z^2-xyz-2 \end{aligned}$$

and how they relate to reps of $\cancel{F_2 \to SL_2\mathbb{R}}$

$$\mathrm{Hom}(F_2, SL_2\mathbb{R}) \subset \mathbb{R}^3$$

③ Understand sketch of proof of Goldman's theorem.

$$\begin{array}{c} \mathrm{Hom}(\pi_1(S_g), PSL_2\mathbb{R}) \\ \downarrow \chi \\ \mathbb{Z} \end{array}$$

<u>Thm</u> Components of $\mathrm{Hom}(\pi_1(S_g), PSL_2\mathbb{R})$

$$\updownarrow$$

$\chi^{-1}(n) \qquad n \in [\chi(g), -\chi(g)] \cap \mathbb{Z}.$

Relative Euler class

- S surface w/ ∂, genus g
- $\pi_1(S) \cong \langle A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_r \mid \prod_{i=1}^{g} [A_i, B_i] \cdot C_1 \cdots C_r = 1 \rangle$
- Given $\rho: \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ w/ $\rho(C_i)$ hyperbolic (or not elliptic)
  - take standard lifts $\widetilde{\rho(C_i)}$ to $\widetilde{SL_2\mathbb{R}}$
  - take arbitrary lifts $\widetilde{\rho(A_i)}, \widetilde{\rho(B_i)}$

Then $\prod [\widetilde{\rho(A_i)}, \widetilde{\rho(B_i)}] \; \widetilde{\rho(C_i)} = z^n \in \widetilde{PSL_2\mathbb{R}}$

where z generates center of $\widetilde{SL_2\mathbb{R}}$

__Defn__ n is the relative Euler number of ρ.

__Thm__ (Goldman) (3.3)

- M compact
- $W(M) = \{ \phi : \pi_1 M \to PSL_2\mathbb{R} \mid \phi(c) \text{ hyperbolic } \forall\, c \subset \partial M \} \subset \mathrm{Hom}(\pi_1 M, \cancel{SL_2\mathbb{R}}\, PSL_2\mathbb{R})$
- $e: W(M) \longrightarrow \mathbb{Z}$ relative Euler.

~~Thm~~ Then connected components of $W(M)$ are $e^{-1}(n)$ for $|n| \leq |\chi(M)|$.

~~App~~ __Cor__
- M closed
- $|e(\phi)| = |\chi(M)| \iff \phi$ discrete faithful.

__Pf__: discrete, faithful is both open and closed in $\mathrm{Hom}(\pi_1 M, G)$ and so its union of components. For discrete faithful $e(\phi) = \pm\chi(M)$ so $DF(\pi_1 M, G)$ must be equal to these 2 comps.

Conj classes in $PSL_2\mathbb{R}$

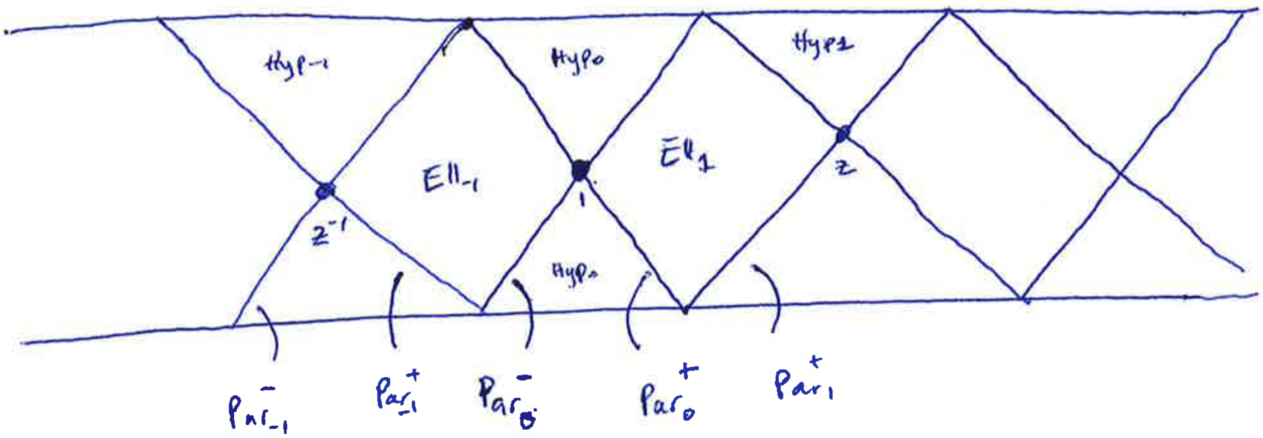
- one for each $r \in (2,\infty)$ $\qquad \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$

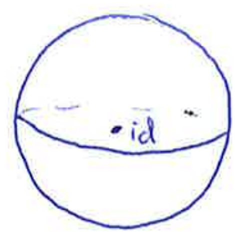
- one for each $r \in (-2,2)$ $\qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad \theta \in (0,\pi)$

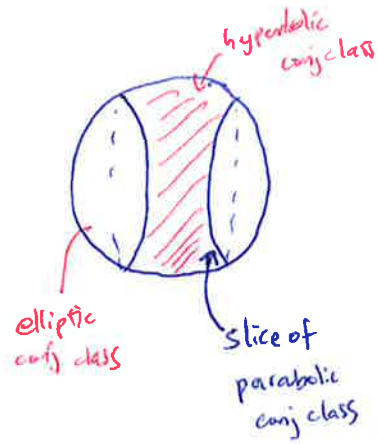
- three for $r = \{+2\}$. $\qquad \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ $\quad t>0$, $t=0$, $t<0$.

KAN and ~~[illegible]~~ conjugacy classes.

$\widetilde{SL_2\mathbb{R}}$







Thm

Let $(x,y,z) \in \mathbb{R}^3$. Then there exists $(X,Y) \in SL_2\mathbb{R} \times SL_2\mathbb{R}$

with $\chi(X,Y) = (x,y,z)$ iff either

- $\kappa(x,y,z) \geq 2$ or
- $(x,y,z) \notin (-2,2)^3$.

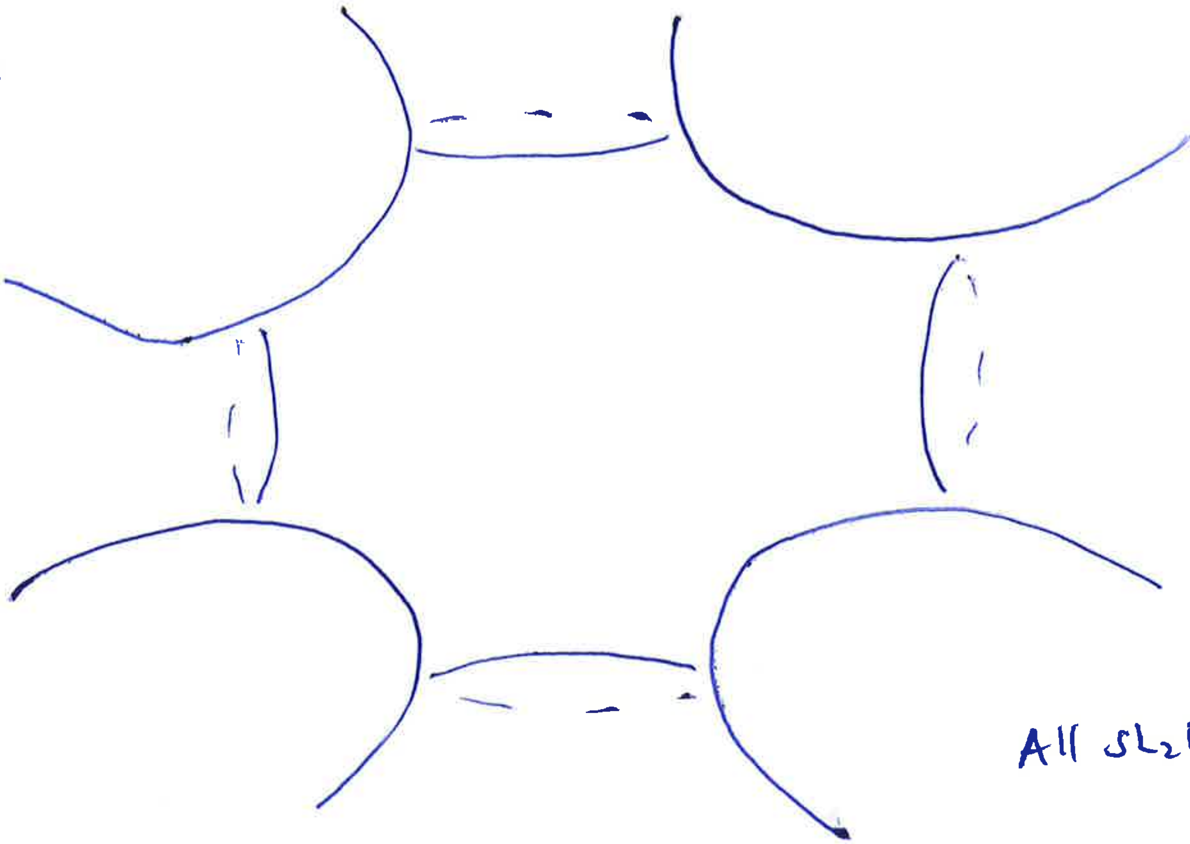
If $\kappa(x,y,z) \neq 2$ $\chi^{-1}(x,y,z)$ is $GL_2\mathbb{R}$ orbit (or 2 $SL_2\mathbb{R}$ orbits)

If $\kappa(x,y,z), |x|, |y|, |z| < 2$ then $\exists$ $(X,Y) \in SU(2) \times SU(2)$

s.t. $\chi(X,Y) = (x,y,z)$.

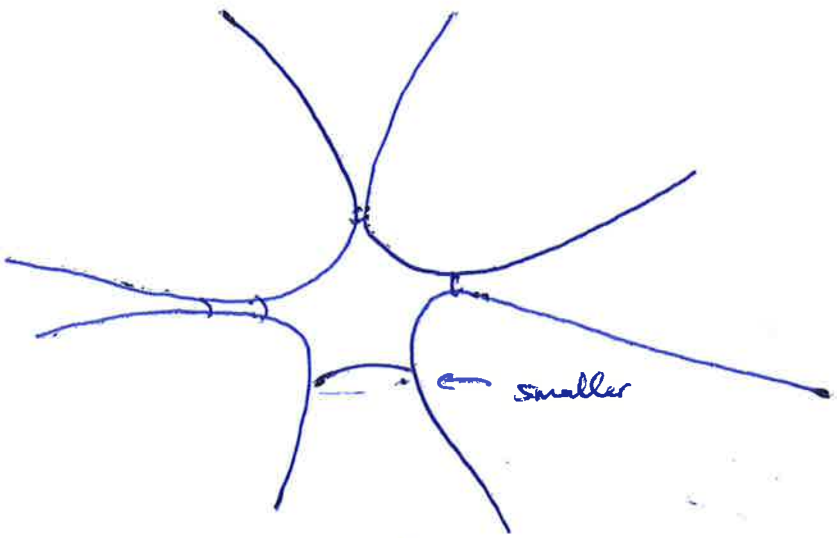
All such pairs (X,Y) are $SU(2)$ conj.

$K^{-1}(100)$



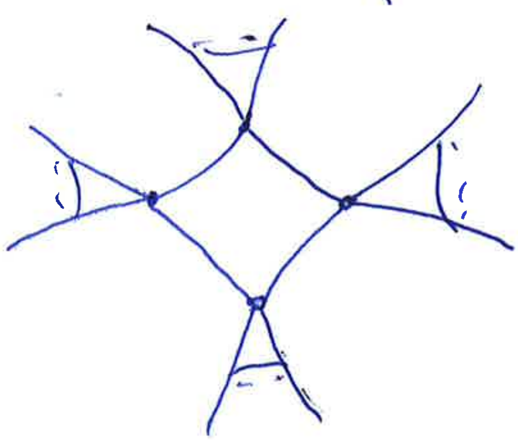
All $SL_2\mathbb{R}$ reps

$K^{-1}(2.0001)$

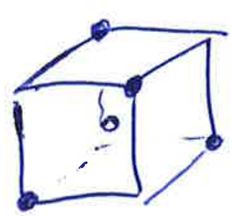


All $SL_2\mathbb{R}$

$K^{-1}(2)$

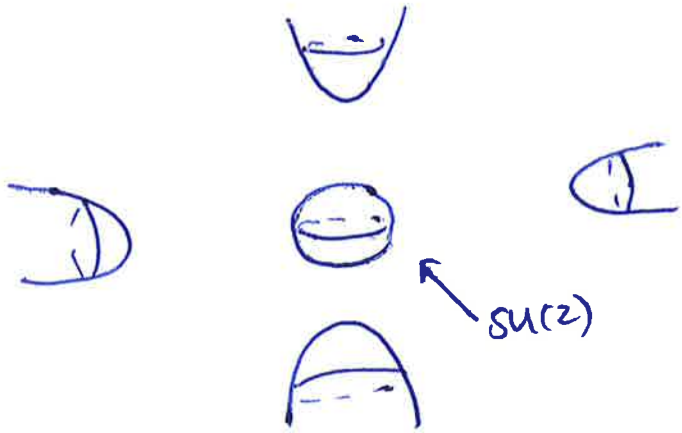


All $SL_2\mathbb{R}$.

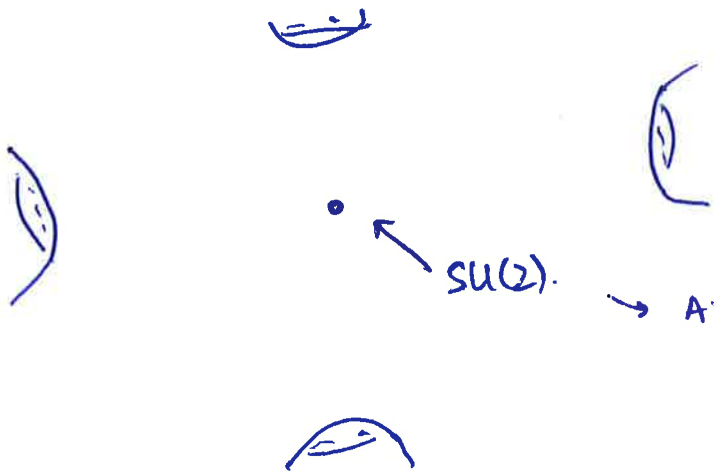


$[-2,2]^3$

$K^{-1}(0)$



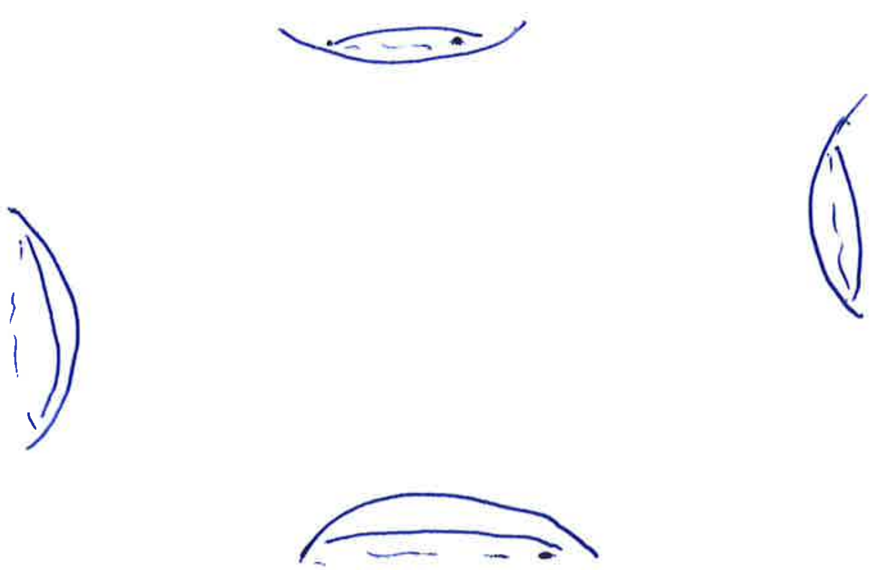
$K^{-1}(-2)$



$\rightarrow A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}$

$K^{-1}(-4)$



Rmk Thm 3.4 gives relative version (ie version for surfaces w/ ∂)

Proof Sketch of Thm 3.3

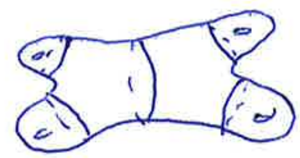
Goal given $\phi, \psi : \pi_1(M) \to PSL_2\mathbb{R}$, with $e(\phi) = e(\psi)$, find path in $W(M)$ connecting them.

- Step 0 Decompose $M = \cup M_i$ into $\chi = -1$ subsurfaces whose dual graph is a tree.

$g=3$
$b=0$

(not )

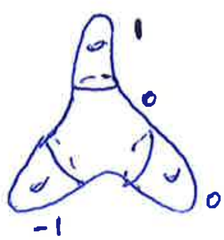
$g=4$
$b=0$



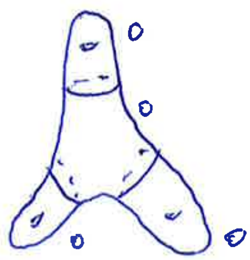
- Step 1 Deform ϕ, ψ so $\phi(C), \psi(C)$ hyperbolic for each $C \subset \partial M_i$

  Then for each i $\phi|_{\pi_1(M_i)}$, $\psi|_{\pi_1(M_i)}$ has relative Euler $\in \{-1, 0, 1\}$

e.g. ϕ'

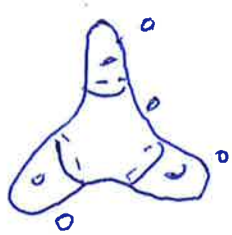


ψ'



- Step 2 Deform ϕ' on ~~genus 2~~ $\chi = -2$ subsurfaces iteratively so that relative Euler #'s match ψ'.

ϕ''



- Step 3 Deform ϕ'' to ψ' on each $\chi = -1$ subsurface.

To-do

I. Prove step1 (9.3, 9.5, 10.1)

II. Prove step2

A. Thm 3.3 for $\chi = -2$ (9.1, 9.2, 9.5)

B. Any relative Euler configs differ by $\chi = -2$ moves.

III. Prove Step3 (7.8)

Main objects in proofs

- trace coords $\quad \chi : SL_2\mathbb{C} \times SL_2\mathbb{C} \longrightarrow \mathbb{C}^3$, $\quad (A, B) \longmapsto (tr(A), tr(B), tr(AB))$

- Commutator polynomial $\quad \kappa : \mathbb{C}^3 \longrightarrow \mathbb{C}$, $\quad (x, y, z) \quad x^2 + y^2 + z^2 - xyz - 2$

- lifted commutator $\quad \tilde{R}_1 : PSL_2\mathbb{R} \times PSL_2\mathbb{R} \longrightarrow \widetilde{SL_2\mathbb{R}}$, $\quad (A, B) \longmapsto \tilde{A} \tilde{B} \tilde{A}^{-1} \tilde{B}^{-1}$.

Thm 7.1 $\quad G = PSL_2\mathbb{R}$

$$\tilde{R}_1 : G \times G \longrightarrow \tilde{G}$$
$$(A,B) \longmapsto [\tilde{A},\tilde{B}]$$

(i) $\operatorname{im} \tilde{R}_1 = \{id\} \cup Ell_{\pm 1} \cup Par_0^{\pm} \cup Hyp_0 \cup Par_{\mp 1}^{\pm} \cup Hyp_{\pm 1}$

(ii) For $C \in \tilde{G}$ $\quad X_1(C) = \{(A,B) \in PSL_2\mathbb{R} \times PSL_2\mathbb{R} \mid [\tilde{A},\tilde{B}] = C\}$ connected.

$Hyp_0 = \{ g \in \tilde{G} : \pi(g)$ hyperbolic, $g \in \operatorname{im}(\exp : \mathfrak{g} \longrightarrow \tilde{G})\}$.

$Hyp_n = Hyp_0\, z^n$

$Ell_1 = \{ g \in \tilde{G}$ conjugate to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \theta \in (0,\pi)\}$.