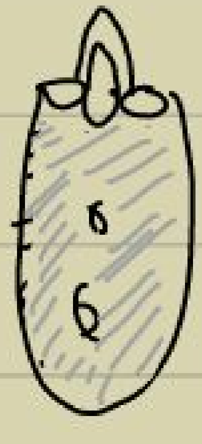
Setup: • $\Sigma_{g,r}$ genus g, r boundary comp

• $\Gamma_{g,r} = \pi_0 \mathrm{Diff}(\Sigma_{g,r} \text{ rel } \partial)$

Stabilization maps:

(i) $\Sigma_{g,r} \xrightarrow{\alpha} \Sigma_{g+1,r-1}$



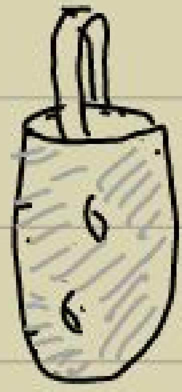
induces

$\mathrm{Diff}(\Sigma_{g,r}) \longrightarrow \mathrm{Diff}(\Sigma_{g+1,r-1})$

and hence $\Gamma_{g,r} \xrightarrow[\alpha]{} \Gamma_{g+1,r-1}$.

Similarly

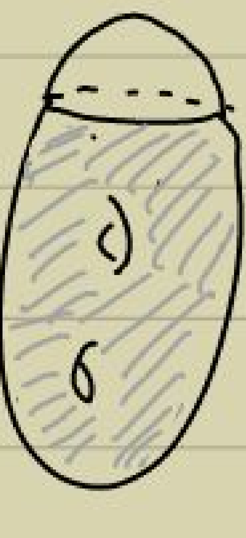
(ii) $\Sigma_{g,r} \xrightarrow{\beta} \Sigma_{g,r+1}$



induces

$\Gamma_{g,r} \xrightarrow{\beta} \Gamma_{g,r+1}$

(iii) $\Sigma_{g,r} \xrightarrow{\delta} \Sigma_{g,r-1}$



induces

$\Gamma_{g,r} \longrightarrow \Gamma_{g,r-1}$

Thm (Harer, Ivanov, Boldsen, Randal-Williams)

$\alpha_* : H_i(\Gamma_{g,r}) \longrightarrow H_i(\Gamma_{g+1,r-1})$ is an iso for $i \leq \frac{2g+1}{3}$

$\beta_* : H_i(\Gamma_{g,r}) \longrightarrow H_i(\Gamma_{g,r+1})$ is an iso for $i \leq 2g/3$.

Rmk: Compose $\beta \circ \alpha$ to get Harer's original stability $\Gamma_{g,r} \longrightarrow \Gamma_{g+1,r}$.

General Approach to homological stability

Let $\cdots \hookrightarrow G_n \hookrightarrow G_{n+1} \hookrightarrow \cdots$ seq of gps.

(I) Construct simplicial complexes $X_n = (X_n)_\bullet$ s.t. $G_n \curvearrowright X_n$ and

(a) $G_n \curvearrowright (X_n)_p$ transitively with stabilizer $\cong G_{n-p-1}$

(b) X_n is highly connected.

II Spectral sequence argument

- Take double cplx $(EG_n)_\bullet \otimes_{\mathbb{Z}G_n} C_\bullet(X_n)$ compute homology in 2 ways to learn about $H_\bullet G_n$.

In our case

$$\Gamma_{g,r} \xrightarrow{\alpha} \Gamma_{g+1,r-1} \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \Gamma_{g+r,0}$$

— AND —

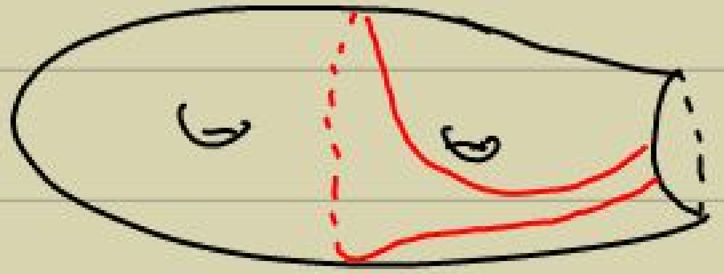
$$\Gamma_{g,0} \to \cdots \to \Gamma_{g,r} \xrightarrow{\beta} \Gamma_{g,r+1} \xrightarrow{\beta} \Gamma_{g,r+2} \xrightarrow{\beta} \cdots$$

For each g,r we will build two sim cplxs $X_{g,r}$ and $Y_{g,r}$.

<u>The arc complex</u>. Fix g,r.

<u>Defn</u> An arc is a map $a: [0,1] \to \Sigma_{g,r}$ with $a(0), a(1) \in \partial\Sigma_{g,r}$

An arc is <u>non-separating</u> if $\text{im}(a)^c$ is connected.



Defn Fix $b_0, b_1 \in \partial \Sigma_{g,r}$ on same boundary comp. Define cplx $X_{g,r}$

$$\text{vertices } V = \left\{ \begin{array}{c} \text{isotopy classes of nonsep} \\ \text{arcs } a \text{ with } a(0) = b_0, a(1) = b_1 \end{array} \right\}$$

$$p\text{-simplices} = \left\{ \begin{array}{l} \text{unordered tuples } (a_0, \ldots, a_p) \in V^{p+1} \\ \exists \text{ reps that are disjoint,} \\ \text{embedded and such that} \\ a_0(0), \ldots, a_p(0) \ \& \\ a_0(1), \ldots, a_p(1) \text{ have} \\ \text{same cyclic ordering on } \partial\Sigma_{g,r} \end{array} \right\}$$

Define $Y_{g,r}$ similarly when b_0, b_1 on different boundary components.

Note $\Gamma_{g,r} \curvearrowright X_{g,r}$ and $Y_{g,r}$. For the sseq we will use 5 properties:

(P1) $\Gamma_{g,r}$ acts transitively on $(X_{g,r})_p$ & $(Y_{g,r})_p$ $\forall p \geq 0$.

(P2) The stabilizer of a p-simplex $\sigma = (a_0, \ldots, a_p)$ is

$\cong \Gamma_{g-p-1,\, r+p+1}$ for $\sigma \in X_{g,r}$.

$\cong \Gamma_{g-p,\, r+p-1}$ for $\sigma \in Y_{g,r}$.

(the important part is that it is a previous group in the series)

(P3,4) $X_{g,r} \xrightarrow{\alpha} X_{g+1,\, r-1}$ induces

β on the stabilizer of $\sigma \in (Y_{g,r})_0$

$$\begin{tikzcd}
\Gamma_{g,r} \arrow[r, "\alpha"] \arrow[d, phantom, "\cup"] & \Gamma_{g+1,\, r-1} \arrow[d, phantom, "\cup"] \\
\mathrm{Stab}_Y(\sigma) \arrow[r, "\beta"] & \mathrm{Stab}_X(\alpha(\sigma))
\end{tikzcd}$$

Moreover, there exists a curve $c \in \Sigma_{g+1,r-1}$ so that

$$\begin{tikzcd}
\Gamma_{g,r} \arrow[r, "\alpha"] \arrow[dr, dashed, "T_c"] & \Gamma_{g+1,r-1} \\
\mathrm{Stab}_Y(\sigma) \arrow[u, hook] \arrow[r, "\beta"] & \mathrm{Stab}_X(\alpha(\sigma)) \arrow[u, hook]
\end{tikzcd}$$

Commutes.

(P5) $X_{g,r}$ & $Y_{g,r}$ are $(g-2)$-connected.

Proof of P1 : Classification of surfaces.

given $(a_0, \dots, a_p)$ and $(a_0', \dots, a_p')$, p-simplices, take two copies $\Sigma_{g,r}$ $\Sigma_{g,r}'$

- cut along a_i, a_i'
- count # ∂ comp, compute χ

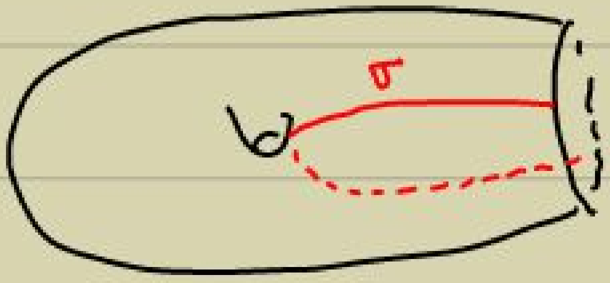
conclude surfaces are diffeo.

Note can ensure preserve ordering.

Proof of P2 We will prove for $X_{g,r}$.

In fact the main difficulties are already present for 0-simplex of $X_{g,r}$

Fix $\sigma = (a_0) \in X_{g,r}$ vertex



Claim. $\text{Stab}_X(\sigma) \cong \Gamma_{g-1,r+1}$

Cut along a_0 to obtain $\Sigma_{g-1,r+1} \rightarrow \Sigma_{g,r}$ and hence $\Gamma_{g-1,r+1} \xrightarrow{\tau} \Gamma_{g,r}$.

Note that $\text{im}(\tau) \subset \text{Stab}_X(\sigma)$.

Step 1: τ surjective.

Let $[\varphi] \in \mathrm{Stab}_X(\sigma)$, where $\varphi : \Sigma_{g,r} \circlearrowleft$ is diffeo.

$[\varphi]\sigma = \sigma \implies \varphi \circ a_0 \sim a_0$ isotopic.

By an isotopy, can ensure $\varphi \circ a_0 = a_0$.

Now cut along a_0 to get $\hat{\varphi} : \Sigma_{g-1,r+1} \circlearrowleft$.

Step 2: τ injective.

Must show the following is impossible.

Let $\varphi \in \Gamma_{g-1,r+1}$. think of φ as a diffeo of $\Sigma_{g,r}$ that fixes the arc a_0.

$[\varphi] \neq 0$ in $\Gamma_{g-1,r+1}$ means $\varphi \not\sim \mathrm{id}$ through diffeos fixing the arc a_0.

if $[\varphi] = 0$ in $\Gamma_{g,r}$ Then $\varphi \sim \mathrm{id}$ through diffeos that move a_0

For example the morphism

$$\pi_0 \mathrm{Diff}(\Sigma_{g,r} \text{ rel } \partial) \longrightarrow \pi_0 \mathrm{Diff}(\Sigma_{g,r})$$

kills DT about ∂-comp.

Pf: Let $S = \Sigma_{g,r}$. $\exists$ fibration

precompose w/ a

$$\begin{tikzcd} \mathrm{Diff}(S \text{ rel } \partial \cup a) \arrow[r] & \mathrm{Diff}(S \text{ rel } \partial) \arrow[r] & \mathrm{Emb}(I, S) \end{tikzcd}$$

On homotopy

$$\begin{tikzcd} \pi_1 \mathrm{Emb}(I,S) \arrow[r, "\mu"] & \pi_0 \mathrm{Diff}(S \text{ rel } \partial \cup a) \arrow[r] \arrow[d, phantom, "\cong" sloped] & \pi_0 \mathrm{Diff}(S \text{ rel } \partial \arrow[d, phantom, "\cong" sloped] \\ & \Gamma_{g-1,r+1} & \Gamma_{g,r} \end{tikzcd}$$

So τ inj $\Longleftrightarrow$ $\mu = 0$. μ is monodromy of the fibration.

By Earle-Eells $\mathrm{Emb}(I, S) \sim *$ so $\mu = 0$.

☑ Pf of P2