Associations of a non-associative algebra

November 6, 2012

Abstract

John Baez describes the octonions as "that crazy old uncle nobody lets out of the attic". In this talk we'll define this eccentric non-associative algebra and describe its connection to Bott periodicity and exceptional Lie groups.

1 Introduction

A normed division algebra A is a

- real vector space
- multiplication (and unit)
- $x \in A \backslash \{0\}$ is invertible
- Norm. $||\cdot|| : A \to \mathbb{R}_{\geq 0}$ with $||ab|| = ||a|| \cdot ||b||$.

Here are all examples:

1. $\mathbb{R}$
2. $\mathbb{C} = \mathbb{R}\{1, i\}$ and the multiplication is determined by $i^2 = -1$.
3. In the 1830s, Hamilton was trying to find a 3-dimensional version, and in the process invented the quaternions
$$\mathbb{H} = \{1,\ i,\ j,\ ij\}$$
with relations $-1 = i^2 = j^2 = (ij)^2$. The quaternions are useful partially because the quaternions of norm 1 form the group SU_2—the universal and double cover of SO_3—and this makes the quaternions nice for studying rotations and angular momentum and quantum physics.
4. John Graves was interested in generalizing Hamilton's construction to obtain normed division algebras in higher dimensions; in particular, he discovered the octonions
$$\mathbb{O} = \{1,\ i,\ j,\ ij,\ \ell,\ i\ell,\ j\ell,\ (ij)\ell\}$$
with some relations like $-1 = i^2 = j^2 = \cdots$ and $(i\ell)(j\ell) = ji$ and $(ij)\ell = -i(j\ell)$.[1] He tried to generalize to still higher dimensions, but got stuck. We'll see why later.

[1] Insert Fano plane description.

2 Octonions and periodicity

2.1 Octonions and Clifford algebras

First we define the clifford algebras:

$$\text{Cliff}(n) = \frac{\langle e_1, \dots, e_n \rangle}{\langle e_i^2 = -1, e_i e_j = -e_j e_i \rangle}$$

Cliff(n) has as a basis $e_{i_1} \cdots e_{i_k}$, where $\{i_1, \dots, i_k\} \subset [n]$ (i.e. dimension 2^n over $\mathbb{R}$). It is an associative algebra. (Normed?) Here are the first few Clifford algebras.

n	Cliff(n)
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}[2]$
5	$\mathbb{C}[4]$
6	$\mathbb{R}[8]$
7	$\mathbb{R}[8] \oplus \mathbb{R}[8]$

Here is the connection to normed division algebras.

Proposition 2.1. *A n-dimensional normed division algebra is a representation of Cliff$(n-1)$.*

[2] This fact is very restraining, since a matrix algebra has a unique irreducible representation and all others are direct sums of this one.

So the octonions are tied to clifford algebras. Here is the relation to periodicity:

$$\boxed{\text{Cliff}(n+8) = \text{Cliff}(n) \otimes \mathbb{R}[16].}$$

This 8-fold periodicity is a manifestation of Bott-periodicity.

2.2 Octonions and Bott periodicity

The infinite orthogonal group is the direct limit $O = \lim O_n$. Bott-periodicity is a statement about the homotopy groups of $O = \lim O_n$.

Theorem 2.2. *For $i \geq 0$, the sequence $\pi_i(O)$ is 8-periodic*

$$\mathbb{Z}/2, \quad \mathbb{Z}/2, \quad 0, \quad \mathbb{Z}, \quad 0, \quad 0, \quad 0, \quad \mathbb{Z}.$$

There are two parts of this theorem. The computation of the small homotopy groups, and the periodicity. We say something about these two parts.

1. $\pi_i(O)$ for $i \leq 7$. Non-trivial elements in these homotopy groups come from division algebras! For each division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, there is a projective line

$$\mathbb{R}P^1 \simeq S^1, \quad \mathbb{C}P^1 \simeq S^2, \quad \mathbb{H}P^1 \simeq S^3, \quad \mathbb{O}P^1 \simeq S^7.$$

[2] I don't think I'll have time to prove this. Maybe just mention that left multiplication defines a representation, and that restricting to the imaginary elements gives a representation of Clifford algebra.

And for each of these spheres S^k there is a canonical rank-k vector bundle

$$E_k \to S^k$$

Any vector bundle over a sphere can be trivialized on the north and south hemi-sphere, so the bundle is determined how the bundle over these hemi-spheres glue, which is a map $S^{k-1} \to \mathrm{GL}_k\mathbb{R}$, well-defined up to homotopy. Then the tautological line bundle gives an element of $\pi_{k-1}(O_k) \to \pi_{k-1}(O)$. We claim these elements are non-trivial, and moreover generate.

2. Periodicity. By general nonsense, any rank-k vector bundle over S^k is the pullback of a "classifying map"

$$f : S^k \to BO_k \to BO.$$

So the bundle $L_{\mathbb{O}} \to S^8$ comes from a map

$$f_{\mathbb{O}} : S^8 \to BO_8 \to BO.$$

Now any map f induces $f \wedge f_{\mathbb{O}} : S^k \wedge S^8 \to BO$, and since $S^k \wedge S^\ell \simeq S^{k+\ell}$, a bundle over S^k and a bundle over S^ℓ determines a bundle over $S^{k+\ell}$.

$$\begin{array}{ccc} \pi_i O & \longrightarrow & \pi_{i+8} O \\ \downarrow \simeq & & \downarrow \\ \pi_{i+1} BO & \longrightarrow & \pi_{i+1+8} BO \end{array}$$

$$(S^{i+1} \xrightarrow{f} BO) \mapsto S^{i+1} \wedge S^8 \xrightarrow{f \wedge f_{\mathbb{O}}} BO.$$

Theorem. This map is an isomorphism for all i. *Hence, the octonions generate Bott periodicity.*

3 Octonions and exceptional Lie algebras

Theorem 3.1. *Every simple Lie algebra is on this list.*

Orthogonal	$\mathfrak{so}(n) = \{x \in \mathbb{R}[n] : x^* = -x, tr(x) = 0\}$
Unitary	$\mathfrak{su}(n) = \{x \in \mathbb{C}[n] : x^* = -x, tr(x) = 0\}$
Symplectic	$\mathfrak{sp}(n) = \{x \in \mathbb{H}[n] : x^* = -x\}$
Exceptional	$\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$

The "corresponding" compact, connected Lie groups are $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, and $\mathrm{SP}(n)$, which act as isometries of $\mathbb{R}P^{n-1}$, $\mathbb{C}P^{n-1}$, and $\mathbb{H}P^{n-1}$. Where do the exceptional groups come from?

Theorem 3.2. *There are spaces* $\mathbb{O}P^2, (\mathbb{O} \otimes \mathbb{C})P^2, (\mathbb{O} \otimes \mathbb{H})P^2, (\mathbb{O} \otimes \mathbb{O})P^2$ *whose isometry group has Lie algebra* $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, *and* $\mathfrak{e}_8$, *respectively.*

One of the hard parts of the theorem is actually defining the spaces. For example, $\mathbb{O}P^1$ cannot be defined as points in $\mathbb{O}^2$ up to scaling by $\mathbb{O}$—it is not an equivalence relation because $\mathbb{O}$ is not-associative.
This theorem is not exactly satisfying. The hardest part is finding the spaces, and for the last three, the space is obtained by quotient by a subgroup.