More Representation Rigidity

- S closed surface genus ≥ 2, $G = \mathrm{Isom}^+\mathbb{H}^2 \cong PSL_2\mathbb{R}$.
  $\Gamma = \pi_1(S)$

<u>Goal</u> ① understand components of $\mathrm{Hom}(\Gamma, G)$

② characterize subspace $\mathrm{HypStruct}(S) \subset \mathrm{Hom}(\Gamma, G)$
$\qquad\qquad \| \qquad$
$\mathrm{DiscreteFaithful}(\Gamma, G)$.

Last Time

- M^n closed hyperbolic $n \geq 3$ $\qquad i : \pi_1 M \xrightarrow{\text{lattice}} \mathrm{Isom}\mathbb{H}^n$

- $\mathrm{Vol} : \mathrm{Hom}(\pi_1 M, \mathrm{Isom}\mathbb{H}^n) \longrightarrow \mathbb{R}$

$$\rho \longmapsto \mathrm{Vol}(\rho)$$

- $\mathrm{im}(\mathrm{Vol}) \subset [-\mathrm{vol}(M), \mathrm{vol}(M)]$
- $\mathrm{Vol}^{-1}(\mathrm{vol}(M))$ = representations conjugate to i.

Today

- S, Γ, G as above
- $\chi : \mathrm{Hom}(\Gamma, G) \longrightarrow \mathbb{Z}$. Euler number
- $\mathrm{im}(\chi) \subset [\chi(S), -\chi(S)]$.
- $\chi^{-1}(\chi(S))$ = ~~discre~~ $\mathrm{HypStruct}(S)$

Euler class & Euler number

- $e \in H^2(G;\mathbb{Z})$

<u>Defn</u> For $\rho : \Gamma \to G$ define $\chi(\rho) = \langle \rho^* e, [S] \rangle$.

Fix $\rho : \Gamma \to G$.

$\boxed{\text{2 ways to define } \rho^* e}$

① <u>Obstruction theory</u> $G \curvearrowright \partial \mathbb{H}^2 \simeq S^1$

ρ defines (flat) circle bundle

$$S^1 \longrightarrow \frac{\tilde{S} \times S^1}{\pi_1 S} \longrightarrow S$$

The obstruction to a section $S \longrightarrow \frac{\tilde{S} \times S^1}{\pi_1 S}$

is a class $\mathcal{O}(\rho) \in H^2(S;\mathbb{Z})$ (the Euler class)

$$\rho^* e := \mathcal{O}(\rho).$$

② <u>Volume</u> Fix $* \in \partial \mathbb{H}^2$

$$\frac{1}{\pi}\mathrm{Vol} : G \times G \times G \longrightarrow \mathbb{Z}$$
$$(g,h,k) \longmapsto \frac{1}{\pi} \cdot \mathrm{Vol}(g*, h*, k*) \in \{0, \pm 1\}$$

$[\frac{1}{\pi}\mathrm{Vol}] \in H^2(G;\mathbb{Z})$ $\qquad \rho^* e := \rho^* [\frac{1}{\pi}\mathrm{Vol}]$.

$$\chi : \mathrm{Hom}(\Gamma, G) \longrightarrow \mathbb{Z}.$$

Information about χ (hence info about flat bundles $S^1 \to E \to S$.) $\longleftrightarrow$ Info about components of $\mathrm{Hom}(\Gamma, G)$.

~~Euler~~ χ bounded (and so Euler # of flat ~~S′-bundles~~ $S^1 \to E \to S$ bounded) $\Longleftarrow$ [G algebraic group & $\mathrm{Hom}(\Gamma, G) \subset G^{2 \cdot \mathrm{genus}(S)}$ algebraic variety $\Rightarrow$ $\mathrm{Hom}(\Gamma, G)$ finitely many components]

<u>Theorem</u> (Milnor, Wood) $\forall \rho \in \mathrm{Hom}(\Gamma, G)$ $|\chi(\rho)| \le |\chi(S)|$.

<u>Thm</u> (Goldman) $\chi(\rho) = \chi(S) \iff \rho$ discrete & faithful.

<u>Thm</u> (Goldman)

Values of χ $\xleftrightarrow{1-1}$ Components of $\mathrm{Hom}(\Gamma, G)$.

$n \in [\chi(S), -\chi(S)] \cap \mathbb{Z} \longmapsto \chi^{-1}(n)$.

Thm (Kneser, 1930)

- S_1, S_2 compact, oriented
- $f: S_1 \longrightarrow S_2$ continuous

Then $|\deg f| \le \dfrac{|\chi(S_1)|}{|\chi(S_2)|}$, with equality $\Leftrightarrow$ $f \sim$ covering map.

Proof $f_*: \pi_1(S_1) \longrightarrow \pi_1(S_2)$.

(Inequality) Choose discrete faithful $i: \pi_1(S_2) \longrightarrow PSL_2\mathbb{R}$.

$$\rho := i \circ f_*: \pi_1(S_1) \longrightarrow PSL_2\mathbb{R}.$$

Milnor-Wood $\Longrightarrow$ $\chi(\rho) \le \chi(S_1)$

$$\chi(\rho) = \langle \rho^* e, [S_1] \rangle = \langle i^* e, f_*[S_1] \rangle$$
$$= \deg f \ \chi(i) = \deg f \ \chi(S_2).$$

so $|\deg f| \le \left| \dfrac{\chi(S_1)}{\chi(S_2)} \right|$.

(Equality) $|\deg f| = \left| \dfrac{\chi(S_1)}{\chi(S_2)} \right| \Leftrightarrow \chi(\rho) = \chi(S_1)$

$\underset{\text{Goldman}}{\Leftrightarrow}$ ρ discrete faithful

$\Leftrightarrow$ f_* injective.

Let $T \xrightarrow[\cancel{f}]{p} S_2$ covering map corresp to $f_*(\pi_1(S_1)) \subset \pi_1(S_2)$.

Fact $\pi_1(S_1) \xrightarrow{\sim} \pi_1(T) \Longrightarrow \exists$ homeo $F: S_1 \to T$.

Claim $p \circ F \sim f$ h.e.

$$\begin{tikzcd} & T \arrow[d, "p"] \\ S_1 \arrow[ur, "F"] \arrow[r, "f"'] & S_2 \end{tikzcd}$$

Pf: Lift to universal cover

$\widetilde{p\circ F}, \tilde{f} : \tilde{S}_1 \longrightarrow \tilde{S}_2$ both f_* equivariant

$\Rightarrow \widetilde{p\circ F} \sim \tilde{f}$ equivariantly homotopic (just do straight line) $\blacksquare$