

More Representation Rigidity

Rigidity Seminar Summer 2014

- S closed surface genus ≥ 2 , $G = \text{Isom}^+ \mathbb{H}^2 \cong \text{PSL}_2 \mathbb{R}$.
 $\Gamma = \pi_1(S)$

Goal ① understand components of $\text{Hom}(\Gamma, G)$

- ② characterize subspace $\text{HypStruct}(S) \subset \text{Hom}(\Gamma, G)$
" Discrete Faithful (Γ, G) .

Last Time

- M^n closed hyperbolic $n \geq 3$ $i: \pi_1 M \xrightarrow{\text{lattice}} \text{Isom} \mathbb{H}^n$

- $\text{Vol}: \text{Hom}(\pi_1 M, \text{Isom} \mathbb{H}^n) \longrightarrow \mathbb{R}$

$$p \longmapsto \text{Vol}(p)$$

- $\text{im}(\text{Vol}) \subset [-\text{vol}(M), \text{vol}(M)]$

- $\text{Vol}^{-1}(\text{vol}(M)) =$ representations conjugate to i .

Today

- S, Γ, G as above

- $\chi: \text{Hom}(\Gamma, G) \longrightarrow \mathbb{Z}$. Euler number

- $\text{im}(\chi) \subset [\chi(S), -\chi(S)]$.

- $\chi^{-1}(\chi(S)) = \text{Discrete HypStruct}(S)$

Euler class & Euler number

- $e \in H^2(G; \mathbb{Z})$

Defn For $p: \Gamma \rightarrow G$ define $\chi(p) = \langle p^*e, [S] \rangle$.

Fix $p: \Gamma \rightarrow G$.

2 ways to define p^*e .

① Obstruction theory $G \simeq \partial H^2 \simeq S^1$

p defines (flat) circle bundle

$$S^1 \rightarrow \frac{\tilde{S} \times S^1}{\pi_1 S} \rightarrow S$$

The obstruction to a section $S \rightarrow \frac{\tilde{S} \times S^1}{\pi_1 S}$
is a class $o(p) \in H^2(S; \mathbb{Z})$ (the Euler class)

$$p^*e := o(p).$$

② Volume Fix $* \in \partial H^2$

$$\frac{1}{\pi} \text{Vol} : G \times G \times G \rightarrow \mathbb{Z}$$
$$(g, h, k) \mapsto \frac{1}{\pi} \cdot \text{Vol}(g^*, h^*, k^*) \in \{0, \pm 1\}$$

$$\left[\frac{1}{\pi} \text{Vol} \right] \in H^2(G; \mathbb{Z}) \quad p^*e := p^* \left[\frac{1}{\pi} \text{Vol} \right].$$

$$\chi: \text{Hom}(\Gamma, G) \longrightarrow \mathbb{Z}$$

Information about χ
(hence info about flat bundles)



Info about components
of $\text{Hom}(\Gamma, G)$.

$$S' \rightarrow E \rightarrow S$$

~~Euler~~

χ bounded

(and so Euler # of
 $S' \rightarrow E \rightarrow S$
flat ~~connections~~
bounded)



G algebraic group $\hat{=}$
 $\text{Hom}(\Gamma, G) \subset G^{2 \cdot \text{gens}(S)}$
algebraic variety
 $\Rightarrow \text{Hom}(\Gamma, G)$ finitely
many components

Theorem (Milnor, Wood) $\forall p \in \text{Hom}(\Gamma, G) \quad |\chi(p)| \leq |\chi(S)|$

Thm (Goldman) $\chi(p) = \chi(S) \iff p$ discrete & faithful.

Thm (Goldman)

Values of χ



Components of $\text{Hom}(\Gamma, G)$.

$$n \in [\chi(S), -\chi(S)] \cap \mathbb{Z}$$



$$\chi^{-1}(n)$$

Thm (Kneser, 1930)

- S_1, S_2 compact, oriented
- $f: S_1 \rightarrow S_2$ continuous

Then $|\deg f| \leq \frac{|\chi(S_1)|}{|\chi(S_2)|}$, with equality $\Leftrightarrow f \sim$ covering map.

Proof $f_*: \pi_1(S_1) \rightarrow \pi_1(S_2)$.

(Inequality) Choose discrete faithful $i: \pi_1(S_2) \rightarrow \text{PSL}_2\mathbb{R}$.

$$p := i \circ f_*: \pi_1(S_1) \rightarrow \text{PSL}_2\mathbb{R}$$

$$\text{Milnor-Wood} \Rightarrow \chi(p) \leq \chi(S_1)$$

$$\begin{aligned} \chi(p) &= \langle p^*e, [S_1] \rangle = \langle i^*e, f_*[S_1] \rangle \\ &= \deg f \cdot \chi(i) = \deg f \chi(S_2). \end{aligned}$$

$$\text{so } |\deg f| \leq \left| \frac{\chi(S_1)}{\chi(S_2)} \right|$$

$$\text{(Equality)} \quad |\deg f| = \left| \frac{\chi(S_1)}{\chi(S_2)} \right| \Leftrightarrow \chi(p) = \chi(S_1)$$

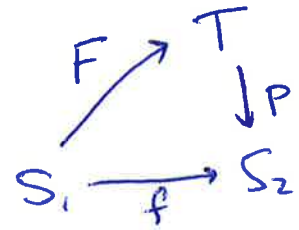
\Leftrightarrow Goldman p discrete faithful

$\Leftrightarrow f_*$ injective.

Let $T \xrightarrow{p} S_2$ covering map correspond to $f_* (\pi_1(S_1)) \cap \pi_1(S_2)$.

Fact $\pi_1(S_1) \xrightarrow{\sim} \pi_1(T) \implies \exists$ homeo $F: S_1 \rightarrow T$.

Claim $p \circ F \sim f$ h.e.



Pf: Lift to universal cover

$\tilde{p} \circ \tilde{F}, \tilde{f}: \tilde{S}_1 \rightarrow \tilde{S}_2$ both f_* equivariant

$\implies \tilde{p} \circ \tilde{F} \sim \tilde{f}$ equivariantly homotopic (just do straight line) \square