References
- Verbitsky papers (Simion's web page)
- Huybrechts "Global torelli theorem for hyperKaehler manifolds"

I. Ergodic Complex Structures

Setup
- X manifold
- $\mathrm{Comp} = \{$ cplx structures on $X\}$
- $\mathrm{Diff}_0 \subset \mathrm{Diff}(X)$ identity component
- ~~Teich~~ $\mathrm{Mod} = \mathrm{Diff}/\mathrm{Diff}_0$ isotopy classes
- $\mathrm{Teich} = \mathrm{Comp}/\mathrm{Diff}_0$ marked moduli space
- $\mathcal{M} = \mathrm{Teich}/\mathrm{Mod} = \mathrm{Comp}/\mathrm{Diff}$ moduli space.

Goal Understand $\mathcal{M}$, Teich, $\mathrm{Mod} \curvearrowright \mathrm{Teich}$

Defn ~~Theorem~~ $I \in \mathrm{Teich}$ is ergodic if $\mathrm{Mod}.(I) \subset \mathrm{Teich}$ is dense

Rmk $\dim_{\mathbb{R}} X = 2 \Rightarrow$ no ergodic cplx structures

($\mathrm{Mod} \curvearrowright \mathrm{Teich} \cong \mathbb{R}^{6g-6}$ properly discontinuously)

Thm (Verbitsky) If X admits hyperKaehler metric, then

$\{$ergodic cplx structures$\} \subset \mathrm{Teich}$ is dense.

<u>Toy Example</u> $X = \mathbb{T}^{2n}$

$\text{Comp} = \{ \text{complex tori } \mathbb{C}^n/\Lambda : \Lambda \subset \mathbb{C}^n \text{ lattice} \}$.

$\text{Teich} = SL_{2n}(\mathbb{R})/SL_n(\mathbb{C})$ $\text{Mod} = SL_{2n}(\mathbb{Z})$.

<u>Claim</u> a.e. $I \in \text{Teich}$ is ergodic.

<u>Pf</u>: <u>Thm</u> (Moore ergodicity) Suppose G simple real Lie group, $H \subset G$ noncpt Lie subgroup, $\Gamma \subset G$ lattice. Then $\Gamma \curvearrowright G/H$ ergodic. (Γ-invar subsets have 0 or full msr)

$\Rightarrow$ Mod $\curvearrowright$ Teich ergodic

$\Rightarrow$ a.e. $I \in \text{Teich}$ has dense orbit $\square$

<u>Rmk</u> Ratner's Theorem gives classification of nonergodic I.

<u>Goal</u> Generalize this argument to HK mflds

$Sp(n)$ holonomy $\leftrightarrow$ $(X_{I,g}, \omega, \Omega)$ ω Kaehler form; Ω holomorphic sympl. form.

<u>Two Steps</u>

1. ~~Under~~ Relate Teich to homogeneous space via period map, Torelli theorems.
2. Relate Mod to arithmetic group via action on $H^2(M;\mathbb{Z})$

II. Teich & Period Map

<u>Case 1</u> X K3 surface $\left(\begin{array}{l} X \text{ HK} \\ \dim_{\mathbb{C}} X = 2 \end{array} \xRightarrow{\text{Kodaira}} \begin{array}{l} X = \mathbb{C}^2/\Lambda \\ \text{or } K3 \end{array} \right)$

- $\left(H^2(X;\mathbb{Z}), \text{cup prod} \right) \simeq (\Lambda, q)$ unimodular, even lattice signature $(3,19)$

- Hodge structure: $I \in \text{Teich} \rightsquigarrow H^2(X;\mathbb{C}) = H^{2,0}(X_I) \oplus H^{1,1}(X_I) \oplus H^{0,2}(X_I)$

$$22 = 1 + 20 + 1$$

$$\begin{aligned} \text{Per}: \text{Teich} &\longrightarrow \mathbb{P}(H^2(X;\mathbb{C})) \\ I &\longmapsto H^{2,0}(X_I). \end{aligned}$$

$$\operatorname{im} \text{Per} \subset \{ [\omega] : q(\omega,\omega) = 0, \ q(\omega,\bar{\omega}) > 0 \} = \Omega_{3,19}$$

For $[\omega] \in \Omega_{3,19}$ $\quad \mathbb{R}\{ \operatorname{Re}(\omega), \operatorname{Im}(\omega) \} \subset H^2(X;\mathbb{R})$ defines positive (wrt q), oriented 2-plane.

$$\Omega_{3,19} \simeq SO(3,19) / SO(2) \times SO(1,19)$$

- <u>Thm</u> Let ~~Suppose~~ X_I, $X_{I'}$ K3 surfaces & $F: H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ isomorphism preserving intersection pairing. Extend F to $H^2(X;\mathbb{C})$

(i) If $F\left(H^{2,0}(X_I) \right) = H^{2,0}(X_{I'})$ then $X_I \simeq X_{I'}$ isomorphic (ie $\exists\, f: X \to X$ diffeo w/ $f^* I = I'$)

(ii) If F sends Kaehler chamber ~~cone~~ $\mathcal{K}_I$ to $\mathcal{K}_{I'}$ then $F = f^*$ for unique isom $f: X_I \to X_{I'}$.

Case 2 X general HK.

Make $H^2(X;\mathbb{Z})$ a lattice

- Beauville - Bogomolov - Fujiki ~~form~~ (integral quadratic) form q

  (i) nondegenerate, signature $(3, b_2-3)$

  (ii) $\exists\, c>0$ s.t. $\forall\, \alpha \in H^2(X;\mathbb{Z})$

$$q(\alpha,\alpha)^n = c\cdot \int_X \alpha^{2n}$$

  (iii) decomp $H^2(X;\mathbb{C}) = (H^{2,0}\oplus H^{0,2})(X) \oplus H^{1,1}(X)$

  orthogonal wrt q.

  If $\underset{H^{2,0}}{\omega\in}$ holomorphic symplectic form $q(\omega,\omega)=0$ $q(\omega,\bar\omega)>0$

~~By (iii) 2 Hodge.~~

Rmks
- q not always unimodular
- unknown if q always even
- unknown which lattices are realized
- unknown if lattice determines diffeo type of X.
  (last time: All K3's diffeo)

By (iii) Hodge structure uniquely determined by $H^{2,0}$.

Define
$$\mathrm{Per}: \mathrm{Teich} \longrightarrow \mathbb{P}(H^2(X;\mathbb{C}))$$
$$I \longmapsto H^{2,0}(X_I).$$

$$\mathrm{Im}(\mathrm{Per}) \subset \{[\omega] : q(\omega,\omega)=0,\ q(\omega,\bar\omega)>0\} = \Omega_{3,b_2-3} \simeq \frac{SO(3,b_2-3)}{SO(2)\times SO(1,b_2-3)}$$

Torelli Theorem

Problem: It is False

<u>Thm</u> (Debarre) $\exists$ HK X_I, $X_{I'}$ non isomorphic w/ isomorphic wt 2 Hodge structures. ($\mathrm{Per}\, I = \mathrm{Per}\, I'$)

Related Problem: Teich is not Hausdorff

<u>Defn</u> Z top space. $x, y \in Z$ <u>inseparable</u> $x \sim y$ if $\forall U \ni x$ $\forall V \ni y$ nbhds $U \cap V \neq \phi$.

<u>Rmk</u> ~~[illegible]~~ $I, I' \in$ Teich. inseparable, $\Rightarrow$ $\mathrm{Per}(I) = \mathrm{Per}(I')$

<u>Thm</u> (Huybrechts)

(i) $I, I' \in$ Teich inseparable $\Longleftrightarrow$ $X_I \sim X_{I'}$ bimeromorphic

($\exists\ f: X_I \to X_{I'}$ multivalued whose graph projects bihol to each factor after removing an analytic subset)

if $X_I, X_{I'}$ varieties bimero $\to$ birational, ie can get from X_I to $X_{I'}$ by blowing $\uparrow/\downarrow$ divisor. Note having divisor is codim 1 condition

(ii) If $\mathrm{Per}(I) = \mathrm{Per}(I')$ and I, I' in same component, then $X_I \sim X_{I'}$ bimeromorphic.

<u>Fix</u> $\mathrm{Teich}_b := \mathrm{Teich}/\sim$ birational Teich sp. $\mathrm{Teich}_b/\mathrm{Mod}$ can still be non Haus (~~[illegible]~~)

smooth, Hausdorff, Cplx mfld.

<u>Thm</u> (Verbitsky) $\mathrm{Per}: \mathrm{Teich}_b \longrightarrow \Omega_{3, b_2-3}$ iso on each ~~component~~ component.

<u>Rmk</u> Fibers of $\mathrm{Teich} \longrightarrow \mathrm{Teich}_b$ corresp to Kahler chambers.

$\Rightarrow$ Countable or finite.

III. Mapping Class Group

$\Gamma = \mathrm{Aut}\left(H^{\bullet}(X;\mathbb{Z}),\ p_1(X), \ldots, p_k(X)\right)$ $\qquad \dim X = 4k.$

$\Gamma_2 = \mathrm{Aut}\left(H^2(X;\mathbb{Z}), q\right)$

<u>Thm</u> (Sullivan) X simply connected, $\dim X \geq 5$, Kaehler. Then

$$\mathrm{Mod}(X) \longrightarrow \Gamma \qquad \text{finite index, finite kernel.}$$

<u>Thm</u> (Verbitsky) X HK

$$\mathrm{Aut}\left(H^{\bullet}(X;\mathbb{R}), p_1(X)\right) \longrightarrow SO(3, b_2-3)$$

surjective, cpt kernel.

$\Rightarrow \mathrm{Mod} \longrightarrow \Gamma_2 \qquad$ finite index, finite kernel.

<u>Warning</u>: Sullivan's theorem needs $\pi_1 X = 0$.

$$0 \longrightarrow \bigoplus_{i=1}^{\infty} \mathbb{Z}/2 \longrightarrow \pi_0 \mathrm{Diff}(\mathbb{T}^k) \longrightarrow SL_k(\mathbb{Z}) \longrightarrow 1$$

IV. Characterizing Ergodic complex Structures

Recall $\mathrm{Pic}(X_I) = H^2(X;\mathbb{Z}) \cap H^{1,1}(X_I)$ rank $\le b_3 - 2$.

Thm (Verbitsky) $I \in$ Teich nonergodic $\iff$ $\mathrm{Pic}(X_I)$ max rank.

($\Rightarrow$ {nonergodic cplx str} countable)

Main ingredient:

Thm (Ratner)

Let G be a real algebraic group (w/ no nontrivial characters)

$H \subset G$ generated by unipotents, $\Gamma \subset G$ arithmetic.

For $g \in G$, $\overline{(\Gamma g) H} \subset \Gamma \backslash G$ is $(\Gamma g) W$ where

W is the smallest real algebraic group containing

$g H g^{-1}$.

Pf sketch of Verbitsky Apply Ratner w/ $G = SO(3,19)$ $H = SO(2) \times SO(1,19)$

$\Gamma = O(H^2(M;\mathbb{Z}), q)$.

Fix $I \in$ Teich nonergodic. Denote $\mathrm{Per}(I) = gH \in G/H$.

I nonerg $\iff$ $\Gamma(gH) \subset G/H$ not dense $\iff$ $(\Gamma g) H \subset \Gamma \backslash G$ not dense.

By Ratner, $\overline{(\Gamma g) H} = (\Gamma g) H$, $g^{-1} \Gamma g \cap H \subset H$ lattice,

and $g H g^{-1}$ defined $/\mathbb{Q}$.

$\iff$ $V_I \in \mathbb{P}(H^2(M;\mathbb{R}))$ rational subspace $V_I = \mathbb{R}\{\mathrm{Re}\,\omega, \mathrm{Im}\,\omega\}$

$\langle [\omega] \rangle = H^{2,0}(X_I)$.

and $(H^{2,0}_{\mathbb{R}} \oplus H^{0,2}_{\mathbb{R}})^{\perp} = H^{1,1}_{\mathbb{R}}$ defined over $\mathbb{Q}$

$\iff$ $\mathrm{Pic}(X_I) = H^{1,1} \cap H^2(X;\mathbb{Z})$ full rank ∎