# Volume Rigidity

## I. Background: Representation spaces and Invariants

- $\Gamma$ f.g. infinite group $\quad \Gamma = \pi_1(\mathbb{H}^n/\Gamma) \quad \Gamma = \pi_1(S)$.
- $G$ Lie
- Goal Understand $\mathrm{Hom}(\Gamma, G)$ or $\mathrm{Hom}(\Gamma, G)/G$.
- Example $\Gamma = \pi_1(S) \quad G = PSL_2\mathbb{R}$.

① (Geometry) $\mathrm{Hom}(\Gamma, G)/G \supset \mathrm{Teich}(S)$ as a component

② (flat Bundles) $\mathrm{Hom}(\Gamma, G)/G$ = projective flat circle bundles up to iso

Note $PSL_2\mathbb{R} \subset \mathrm{Homeo}^+ S^1$

Given $\rho : \Gamma \to G$ form $S^1 \to \dfrac{\tilde{S} \times S^1}{\Gamma} \longrightarrow \dfrac{\tilde{S}}{\Gamma}$

circle bundle w/ monodromy in $PSL_2\mathbb{R}$.

Swap

③ (Group actions) $\mathrm{Hom}(\Gamma, G)/G$ = projective actions on $S^1$

- Invariants For $\alpha \in H^*(G)$ get invariant

$$\mathrm{Hom}(\Gamma, G) \longrightarrow H^*(\Gamma)$$

$$\rho \longmapsto \rho^*(\alpha).$$

Examples

① Euler class $\quad e \in H^2(PSL_2\mathbb{R}; \mathbb{Z}) \quad eu(\rho) = |\langle \rho^* e, [S] \rangle|$

② Kaehler class $\quad \kappa \in H^2(SU(n,1); \mathbb{Z}) \quad \tau(\rho) = |\langle \rho^* \kappa, [S] \rangle|$

③ Volume class $\quad \omega \in H^n_{cts}(SO(n,1); \mathbb{R}). \quad vol(\rho) = |\langle \rho^* \omega, [M] \rangle|$

# II. Representation Rigidity

<u>Thm1</u> (Goldman) $i : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ hyperbolic structure.

$\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ anything.

Then $|eu(\rho)| \le |eu(i)| = |\chi(S)|$

with equality $\Longleftrightarrow$ $\rho$ discrete faithful.

<u>Thm2</u> (Toledo) $i : \pi_1(S) \longrightarrow SU(1,1) \longrightarrow SU(n,1)$

$\rho : \pi_1(S) \longrightarrow SU(n,1)$ anything.

Then $\tau(\rho) \le \tau(i) = 2\pi\,\chi(S)$

with equality $\Longleftrightarrow$ $\exists\ \rho_1 : \pi_1(S) \to SU(1,1)$ and $\rho_2 : \pi_1(S) \longrightarrow U(n-1)$ so

$\rho$ conj to $\begin{pmatrix} (\det \rho_2)^{-1/2}\rho_1 & \\ & \rho_2 \end{pmatrix}$

<u>Thm 3</u> (Bucher-Burger-Iozzi) $i : \Gamma \hookrightarrow \mathrm{Isom}\,\mathbb{H}^n$ cocpt lattice.

$\rho : \Gamma \longrightarrow \mathrm{Isom}\,\mathbb{H}^n$ anything.

Then $\mathrm{Vol}(\rho) \le \mathrm{Vol}(i) = \mathrm{Vol}(M)$.

if $n \ge 3$ equality $\Longleftrightarrow$ $\rho$ conjugate to $i$.

## Remarks

1. $PSL_2\mathbb{R} \doteq SU(1,1)$ Thm 2 generalizes Thm 1.
2. All proved w/ bounded cohomology.
3. Thm 3 $\Longrightarrow$ Mostow & Thurston's generalization.

__Thurston__ $M, N$ hyperbolic $f : N \longrightarrow M$

$$\deg f \leq \frac{Vol(N)}{Vol(M)}.$$

with equality $\Leftrightarrow$ isometric covering.

~~Pf (deg f) Vol(M) $\Gamma = \pi_1 M$ $\Gamma' = \pi_1 N$~~

~~Pf: $\Gamma' \to$ Isom $H^n$ $f \to$ Isom $H^n$~~

~~(deg f) Vol(M) = ... Vol(N) = Vol(N).~~

"All roads lead to Mostow"

# III. Volume of a representation

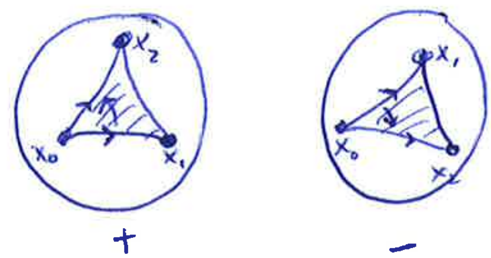
- $M = \mathbb{H}^n/\Gamma$ closed hyperbolic
- $\rho : \Gamma \to \mathrm{Isom}\,\mathbb{H}^n$ anything
- $G := \mathrm{Isom}\,\mathbb{H}^n$

<u>Defn</u>

$$V : (\overline{\mathbb{H}^n})^{n+1} \longrightarrow \mathbb{R}$$

$$(x_0, \dots, x_n) \longmapsto \mathrm{Vol}(\mathrm{ConvHull}(x_0, \dots, x_n)).$$

<u>Rmk</u> signed volume.



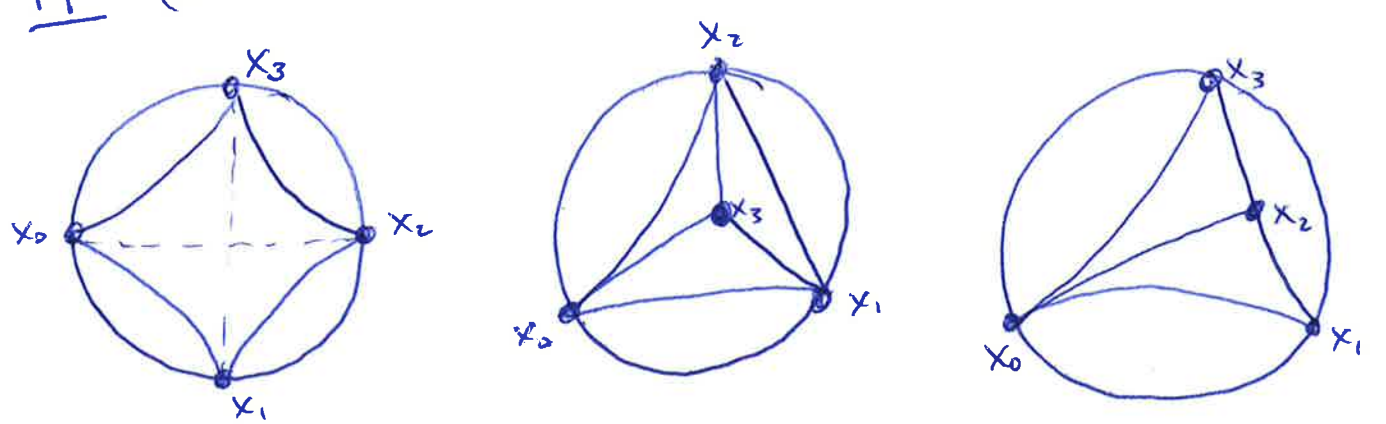

<u>Properties</u>

① $\mathrm{Isom}\,\mathbb{H}^n$ invariant if $g \in G$ $V(gx_0, \dots, gx_n) = V(x_0, \dots, x_n)$.

② $V$ is a cocycle: if $(x_0, \dots, x_{n+1}) \in (\overline{\mathbb{H}^n})^{n+2}$ then

$$0 = \delta V(x_0, \dots, x_{n+1}) = \sum (-1)^i V(x_0, \dots, \hat{x}_i, \dots, x_{n+1}).$$

<u>Pf</u> $(n=2)$



Fix basepoint $* \in \mathbb{H}^n$

Evaluation $\omega : G^{n+1} \xrightarrow{\varepsilon} (\mathbb{H}^n)^{n+1} \xrightarrow{V} \mathbb{R}$

$(g_0, \dots, g_n) \mapsto (g_0 *, \dots, g_n *) \longmapsto V(g_0 *, \dots, g_n *)$.

defines n-cocycle on $G$. $\rightsquigarrow \omega \in H^n_{cts}(G;\mathbb{R})$

"Volume cocycle"

<u>Defn</u> For $\rho : \Gamma \to G$ $\quad \rho^* : H^n(G) \longrightarrow H^n(\Gamma) \simeq H^n(M)$

$$\mathrm{Vol}(\rho) = \langle \rho^* \omega, [M] \rangle.$$

<u>Proof of Thurston's thm</u> $M, N$ hyperbolic $f : N \to M$.

$f : \pi_1 N \to \pi_1 M$.

- $M = \mathbb{H}^n/\Gamma \quad N = \mathbb{H}^n/\Gamma'$
- $f : N \to M$ induces $\rho : \Gamma' \xrightarrow{f} \Gamma \xhookrightarrow{i} \mathrm{Isom}\,\mathbb{H}^n$

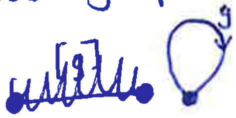
$$\begin{aligned}
\mathrm{Vol}(N) \geq \mathrm{Vol}(\rho) = \langle \rho^* \omega, [N] \rangle &= \langle f^* i^* \omega, [N] \rangle \\
&= \langle i^* \omega, f_*[N] \rangle \\
&= \langle i^* \omega, [M] \rangle \cdot \deg f \\
&= \mathrm{Vol}(i) \deg f \\
&= \mathrm{Vol}(M) \cdot \deg f
\end{aligned}$$

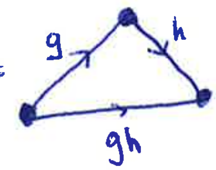
equality $\Leftrightarrow$ $\rho$ lattice $\Leftrightarrow$ $\Gamma' < \Gamma$ finite index subgp $\Leftrightarrow$ $f \sim$ isometric cover.

Claim $\mathrm{Vol}(i) = \mathrm{Vol}(M)$. $\quad i : \Gamma \longrightarrow \mathrm{Isom}\,\mathbb{H}^n$ cocompact, lattice

Pf ($n=2$)

Recall group cohomology $\qquad C_n(G) = \mathbb{Z}\{G^n\}$

$[g] =$  $\quad \partial[g] = 0.$

$(g|h) =$ 

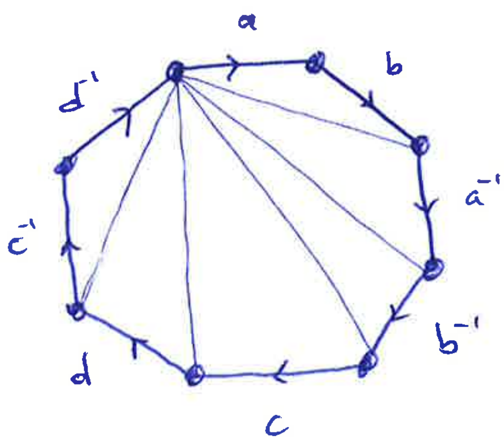


$\partial[g|h] = [h] - [gh] + [g].$

$\partial(g,h) = (h) - (gh) + (g)$

$M = S_2$ genus 2

Fundamental domain





$\subset \mathbb{H}^2$

$z = [a|b] + [ab|a^{-1}] + [aba^{-1}, b^{-1}] + [[a,b]|c] + [[a,b]c|d]$

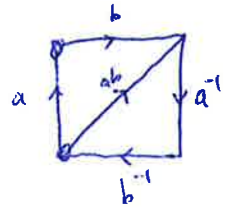
$\quad + [[a,b]cd|c^{-1}] \quad + \text{ degenerate simplices}$

$\langle \omega, (a,b) \rangle = \mathrm{Vol}\Big($  $\Big)$

$$\mathrm{Vol}(i) = \langle i^*\omega, [M] \rangle = \langle i^*\omega, z \rangle = \langle \omega, i_* z \rangle$$

$$= \mathrm{Vol}(\text{fund dom for } \Gamma)$$

$$= \mathrm{Vol}(M).$$




$$\partial\left[(a,b) + (ab,a^{-1}) - (a,a^{-1}) - (e,e)\right]$$

$$= \cancel{(a)} + \cancel{(b)} - \cancel{(ab)} + \cancel{(ab)} + \cancel{(a^{-1})} - \cancel{(aba^{-1})}$$

$$- \cancel{(a)} - \cancel{(a^{-1})} + \cancel{(e)} \quad - \cancel{(e)} - \cancel{(e)} + \cancel{(e)} \qquad = 0$$