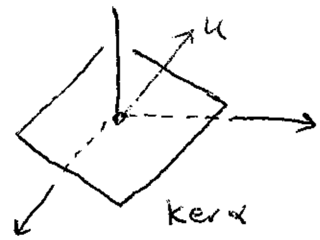
Forms

Defn $\phi \in \Lambda^k(\mathbb{R}^n)$ is called a k-form.

Forms on $\mathbb{R}^3$ $\quad \mathbb{R}^3 = \mathbb{R}\{e_1, e_2, e_3\}$ w/ inner product $\langle \cdot , \cdot \rangle$

$$\langle e_i, e_j \rangle = \delta_{ij}$$

- 1-forms. Given $u \in \mathbb{R}^3$ define $\alpha_u : \mathbb{R}^3 \longrightarrow \mathbb{R}$
$$v \longmapsto \langle u, v \rangle.$$



Every 1-form arises in this way:

Given $\alpha \in \Lambda^1(\mathbb{R}^3)$

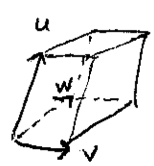
$\exists !\ u$ so that $u \perp \ker(\alpha)$

and $\alpha(u) = \|u\|^2$. Then $\alpha(v) = \langle u, v \rangle$.

$$\Rightarrow \quad \phi : \mathbb{R}^3 \longrightarrow \Lambda^1(\mathbb{R}^3) \quad \text{bijection.}$$
$$u \longmapsto \alpha_u$$

Define $dx_i := \phi(e_i)$. basis for $\Lambda^1(\mathbb{R}^3)$

- 2-forms. Fix $u \in \mathbb{R}^3$ define $\beta_u : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$\beta_u(v, w) = \mathrm{vol}\Bigg($  $\Bigg) = \langle u, v \times w \rangle.$

Exercise every $\beta \in \Lambda^2(\mathbb{R}^3)$ arises in this way. (special feature of $\mathbb{R}^3$)

$$\Rightarrow \quad \psi : \mathbb{R}^3 \longrightarrow \Lambda^2(\mathbb{R}^3) \quad \text{bijection.}$$
$$u \longmapsto \beta_u$$

$\psi(e_1) = dx_2 \wedge dx_3$

$\psi(e_2) = dx_3 \wedge dx_1$

$\psi(e_3) = dx_1 \wedge dx_2$.

- 3-forms For $\omega \in \Lambda^3(\mathbb{R}^3)$ $\exists\, r \in \mathbb{R}$ so that

$$\omega(u,v,w) = r \cdot \det(u|v|w) = r \cdot \langle u, v \times w \rangle.$$

<u>Summary</u>

1-form $\longleftrightarrow$ vector

2-form $\longleftrightarrow$ vector

3-form $\longleftrightarrow$ scalar.

<u>Differential form</u>

<u>Defn</u> A <u>diff'l k-form</u> on $\mathbb{R}^n$ is a choice of k-form for each point of $\mathbb{R}^n$

- $k=0$ $\quad \Omega^0(\mathbb{R}^n) = \{ f : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ smooth} \}.$
- $\cancel{\Omega^k(\mathbb{R}^n)}$ $\quad \Omega^k(\mathbb{R}^n) = \Omega^0(\mathbb{R}^n) \otimes \Lambda^k(\mathbb{R}^n).$
  $k \geq 1$

<u>On $\mathbb{R}^3$</u>

- Every $\alpha \in \Omega^1(\mathbb{R}^3)$ has form $\quad \alpha = A\,dx_1 + B\,dx_2 + C\,dx_3$
  $A, B, C : \mathbb{R}^3 \xrightarrow[\text{smooth}]{} \mathbb{R}$
- Every $\beta \in \Omega^2(\mathbb{R}^3)$ has form $\quad \beta = P\,dx_1 \wedge dx_2 + Q\,dx_1 \wedge dx_3 + R\,dx_2 \wedge dx_3$
  $P, Q, R : \mathbb{R}^3 \xrightarrow[\text{smooth}]{} \mathbb{R}$

<u>Operations</u> For $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ $\quad dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

(i) wedge product $\quad (f\,dx_I) \wedge (g\,dx_J) = (fg)\,dx_I \wedge dx_J.$

(ii) exterior diff'l

$$d: \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$$
$$f \longmapsto \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

$$d(f\,dx_I) = df \wedge dx_I.$$

<u>de Rham complex</u> $\mathfrak{X}(\mathbb{R}^3)$ = vector fields = $\{ u : \mathbb{R}^3 \to \mathbb{R}^3 \}$

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \\
\Omega^0(\mathbb{R}^3) \arrow[u, equal] \arrow[r, "\mathrm{grad}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[u, "\simeq"'] \arrow[r, "\mathrm{curl}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[u, "\simeq"'] \arrow[r, "\mathrm{div}"] & \Omega^0(\mathbb{R}^3) \arrow[u, "\simeq"']
\end{tikzcd}$$

$d^2 = 0$

$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

$\mathrm{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$

$\mathrm{curl}(u) = \nabla \times u$

$\mathrm{div}(u) = \nabla \cdot u$

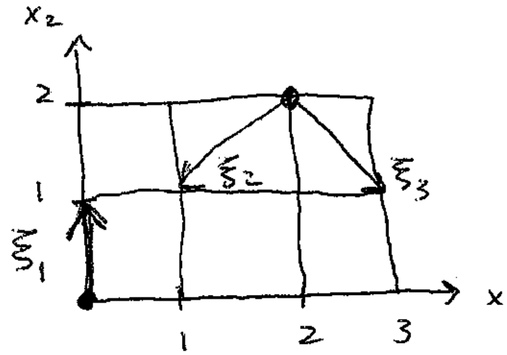
so exterior diff'l generalizes classical vector calculus in $\mathbb{R}^3$.

Problem 1 Calculate the value of each ~~the~~ forms on each vector

$\omega_1 = dx_1$

$\omega_2 = x_1\, dx_2$

$\omega_3 = dr^2 \quad (r^2 = x_1^2 + x_2^2)$



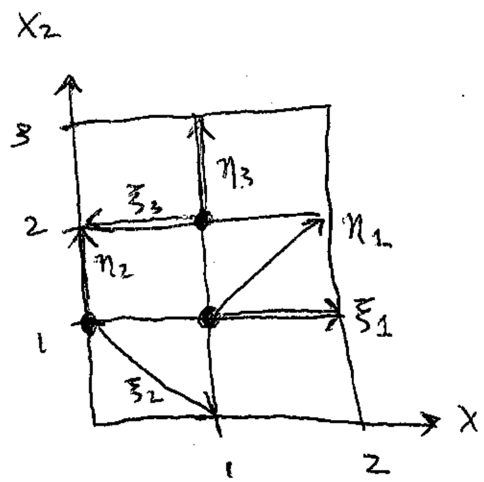
~~on the vectors~~

Problem 2 Calculate the value of each form on each pair of vectors. (ξ_i, η_i)

$\omega_1 = dx_1 \wedge dx_2$

$\omega_2 = x_1\, dx_1 \wedge dx_2 - x_2\, dx_2 \wedge dx_1$

$\omega_3 = r\,dr \wedge d\varphi \quad \left(\begin{array}{l} x_1 = r\cos\varphi \\ x_2 = r\sin\varphi \end{array}\right)$



Problem 3 Show that every differential 1-form on $\mathbb{R}$ is the differential of some function.

Problem 4 A vector field in the plane $\mathbb{R}^2$ is called area preserving if its divergence is 0. (i) Show that the vector field obtained by rotating by $90°$ the gradient of a smooth function $f: \mathbb{R}^2 \to \mathbb{R}$ preserves area.

(ii) Show every area preserving vector field is of this form.