Mostow Rigidity and Bounded Cohomology

I. Definitions and Statement of Main Thm

- X top sp
- $C_k(X)$ singular chain cplx
  has ℓ^1-norm: if $z = \sum c_i \sigma_i$ $\quad c_i \in \mathbb{R}$, $\sigma_i : \Delta^k \to X$,
  $$|z|_1 = \sum |c_i|.$$
- $C^k(X) \equiv \mathrm{Hom}(C_k(X), \mathbb{R})$ has dual ℓ^∞-norm
  for $f : C_k(X) \longrightarrow \mathbb{R}$
  $$|f|_\infty = \sup_{\sigma : \Delta^k \to X} f(\sigma).$$
- $\rightsquigarrow$ seminorms on $H_k(X)$, $H^k(X)$. E.g. for $\alpha \in H_k(X)$
  $$\|\alpha\| = \inf_{[z] = \alpha} |z|_1$$
- For N compact manifold, the <u>simplicial volume</u> is the norm of fundamental class $[N]$.
- <u>Thm</u> (Gromov) Let M closed, hyperbolic. Then
  $$\|M\| = \frac{\mathrm{vol}(M)}{v_n},$$
  where v_n = max volume of ideal n-simplex in $\mathbb{H}^n$.

Applications

① For $g \geq 2$ $\|\Sigma_g\| = \frac{-2\pi \chi(\Sigma_g)}{\pi} = -2\chi(\Sigma_g)$

(Gauss-Bonnet)

② Mostow rigidity: Let $M \sim N$ h.e. hyperbolic

(i) Thm $\Rightarrow$ $\mathrm{vol}(M) = \mathrm{vol}(N)$

(ii) $\Rightarrow$ the map $f: \overline{\mathbb{H}^n} \longrightarrow \overline{\mathbb{H}^n}$ sends a.e. ideal n-simplex of max volume to a ideal n-simplex of max volume.

(iii) $\Rightarrow$ $f|_\partial : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$ equals g conformal a.e.

Rmk Gromov's proof of Thm ⋮ (ii) use measure homology, but can also be proved using bounded cohomology.

II. Continuous, bounded cohomology

- G top gp.
- $C_c^k(G) = \{ f: G^{k+1} \longrightarrow \mathbb{R} \text{ cts} \}$

  chain complex with $C_c^k(G) \xrightarrow{\delta} C_c^{k+1}(G)$

  $$\delta f(g_0, \ldots, g_{k+1}) = \sum (-1)^i f(g_0, \ldots, \hat{g_i}, \ldots, g_{k+1})$$

- $H_c^k(G) = \dfrac{\ker [C_c^k \longrightarrow C_c^{k+1}]}{\mathrm{im}[C_c^{k-1} \longrightarrow C_c^k]}$

- Similarly $C^k_{c,b}(G) = \{ f : G^{k+1} \to \mathbb{R} \ \substack{c+s \\ bdd} \}$ gives continuous, bounded cohomology groups $H^k_{c,b}(G)$

- <u>Theorem</u> (van Est) Let G connected Lie group and $K < G$ max cpt. Fix $x \in G/K$. The map

$$\Omega^k(G/K)^G \longrightarrow H^k_c(G)$$
$$\alpha \longmapsto \left[(g_0, \dots, g_k) \mapsto \int_{\Delta(g_0 x, \dots, g_k x)} \alpha \right]$$

is an isomorphism.

III. Proof of Main Thm

- Let M^n closed hyperbolic, $\Gamma = \pi_1 M$
- $G = \mathrm{Isom}^+ \mathbb{H}^n$, $\overline{G} = \mathrm{Isom}\, \mathbb{H}^n$
- Proof Outline

<u>Step 1</u> Show $\|M\| = \dfrac{\mathrm{Vol}(M)}{\|\omega_M\|}$ where $\omega_M \in H^n_{dR}(M)$ volume form.

<u>Step 2</u> Show $\|\omega_{\mathbb{H}^n}\| = v_n$ where $\omega_{\mathbb{H}^n} \in \Omega^n(\mathbb{H}^n)^G \simeq H^n_c(G)$ volume form.

<u>Step 3</u> Show $\|\omega_M\| = \|\omega_{\mathbb{H}^n}\|$.

Step 1 Let $\varphi \in H^n(M)$ so that $\langle \varphi, [M] \rangle = 1$.

Claim $\|M\| = \frac{1}{\|\varphi\|}$. $\Rightarrow$ (since $\langle \omega_M, [M] \rangle = \mathrm{vol}\, M$

by defn, $\omega_M = \mathrm{vol}(M) \cdot \varphi$

$\rightarrow \frac{\mathrm{Vol}(M)}{\|\omega_M\|} = \frac{1}{\|\varphi\|} = \|M\|$ ✓)

• Duality $H^n(M) \times H_n(M) \longrightarrow \mathbb{R}$

$\Rightarrow 1 = \langle \varphi, [M] \rangle \leq \|\varphi\| \cdot \|M\|$

$\Rightarrow \|M\| \geq \frac{1}{\|\varphi\|}$.

• For the other inequality we want a cocycle $f \in C^n(M)$ with $[f] = \varphi$ and $|f|_\infty \leq \frac{1}{\|M\|}$, ie we want $f : C_n(M) \rightarrow \mathbb{R}$ st

(i) $f\big|_{B_n(M)} \equiv 0$ (f cocycle)

(ii) Fix $z \in C_n(M)$ $[z] = [M]$. Want $f(z) = 1$. ($[f] = \varphi$)

(iii) $|f|_\infty \leq \frac{1}{|z|}$ ($\Rightarrow \|\varphi\| \leq |f|_\infty \leq \frac{1}{|z|_1} \leq \frac{1}{\|M\|}$)

Such an f exists by Hahn-Banach.

Step 2 Compute $\|\omega_{\mathbb{H}^n}\| = v_n$.

Two parts

(i) upper bound : explicit cocycle

(ii) lower bound : explicit resolution

(i) van Est isomorphism gives explicit cocycle.

$$\omega(g_0, \ldots, g_n) = \int_{\Delta(g_0 x, \ldots, g_n x)} \omega_{\mathbb{H}^n}$$, and clearly

$$|\omega|_\infty = v_n, \text{ so } \quad \|\omega_{\mathbb{H}^n}\| \le |\omega|_\infty = v_n.$$

(ii) To compute $\|\omega_{\mathbb{H}^n}\|$ it would suffice to understand

$$\eta : H^n_{c,b}(G) \longrightarrow H^n_c(G)$$

(in particular if η is an iso the we're reduced to computing norm of unique preimage)

Open Problem: is η an isomorphism?

Work-around: replace G by $\overline{G} = \mathrm{Isom}\,\mathbb{H}^n$

replace $\mathbb{R}$ by $\mathbb{R}_\varepsilon$ where

$$\varepsilon : \overline{G} \longrightarrow \mathbb{Z}/2 \subset \mathrm{Aut}(\mathbb{R})$$

$$g \longmapsto \begin{cases} 1 & g \text{ or pres} \\ -1 & g \text{ or rev.} \end{cases}$$

Summary:

$$\begin{tikzcd}
\omega^b_{\mathbb{H}^n} \in H^n_{c,b}(G) \arrow[d] & H^n_{c,b}(\overline{G};\mathbb{R}_\varepsilon) \ni \overline{\omega}^b_{\mathbb{H}^n} \arrow[d] \\
\omega_{\mathbb{H}^n} \in H^n_c(G) & H^n_c(\overline{G};\mathbb{R}_\varepsilon) \ni \overline{\omega}_{\mathbb{H}^n} \arrow[l, "\cong"', "\text{isometric}"]
\end{tikzcd}$$

ω extends to $\overline{\omega} : (\overline{G})^{n+1} \longrightarrow \mathbb{R}$, but

$$\overline{\omega}(gg_0, \dots, gg_n) = \varepsilon(g)\, \overline{\omega}(g_0, \dots, g_n),$$

which says that $\overline{\omega}$ is not $\overline{G}$-invar, but is $\overline{G}$-equivariant as a map $(\overline{G})^{n+1} \longrightarrow \mathbb{R}_\varepsilon$.

<u>Fact</u> $0 \to \mathbb{R} \to C_{c,(b)}(\mathbb{H}^n, \mathbb{R}_\varepsilon) \to C_{c,(b)}((\mathbb{H}^n)^2, \mathbb{R}_\varepsilon) \to \cdots$ is an injective resolution computing $H^*_{c,(b)}(\overline{G};\mathbb{R}_\varepsilon)$.

<u>Note</u> $C_{c,(b)}((\mathbb{H}^n)^{k+1}, \mathbb{R}_\varepsilon)^G = 0$ for $k < n$.

Pf: Fix $f \in C_c((\mathbb{H}^n)^{k+1}, \mathbb{R}_\varepsilon)^G$ and $(x_0, \dots, x_k) \in (\mathbb{H}^n)^{k+1}$. Take $g \in \overline{G}$ reflection in hyperplane containing $x_0, \dots, x_k$. Then

$\mathbb{H}^3$

$$f(x_0, \dots, x_k) = f(gx_0, \dots, gx_k) = \varepsilon(g)\, f(x_0, \dots, x_k) \implies f \equiv 0.$$

So

$$H^n_{c,b}(\bar{G};\mathbb{R}_\varepsilon) = \ker\left[C_{c,b}((\mathbb{H}^n)^{n+1},\mathbb{R}_\varepsilon) \to C_{c,b}((\mathbb{H}^n)^{n+2},\mathbb{R}_\varepsilon)\right]$$

$$\downarrow$$

$$H^n_c(\bar{G};\mathbb{R}_\varepsilon) = \ker\left[C_c((\mathbb{H}^n)^{n+1},\mathbb{R}_\varepsilon) \longrightarrow C_c((\mathbb{H}^n)^{n+2},\mathbb{R}_\varepsilon)\right]$$

since $H^n_{c,b}(\bar{G};\mathbb{R}_\varepsilon)$ is nontrivial and a subspace of $H^n_c(\bar{G};\mathbb{R}_\varepsilon) \cong H^n_c(G) \cong \mathbb{R}$, the comparison map $H^n_{c,b} \longrightarrow H^n_c$ is an iso.

$$\Longrightarrow \quad \begin{array}{l} \|\bar{\omega}_{\mathbb{H}^n}\| = \|\bar{\omega}^b_{\mathbb{H}^n}\| = |\bar{\omega}^b| = v_n. \\ \quad \| \\ \|\omega_{\mathbb{H}^n}\| \end{array}$$

✓

Step3 Show $\|\omega_M\| = \|\omega_{\mathbb{H}^n}\|$.

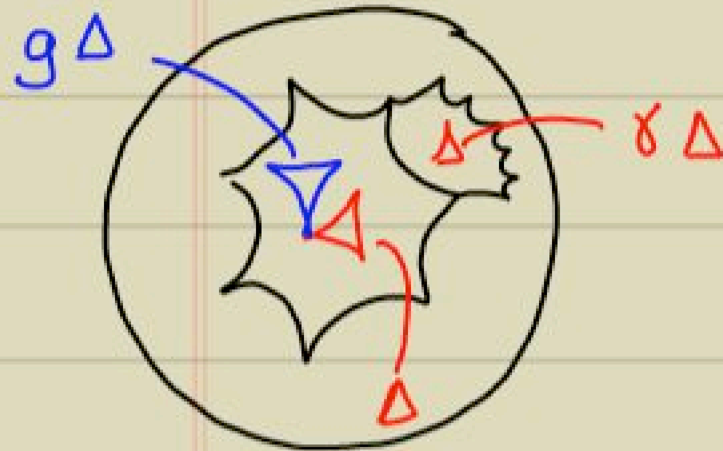
Consider the composition

$$H^n_c(G) \xrightarrow{\rho^*} H^n(\Gamma) \xrightarrow{\tau} H^n_c(G).$$

ρ^* induced by $\rho : \Gamma \longrightarrow G$ and τ is transfer, realized on chain level by

$$C_c\left(\left(\mathbb{H}^n\right)^{n+1}\right)^{\Gamma} \longrightarrow C_c\left(\left(\mathbb{H}^n\right)^{n+1}\right)^{G}$$

$$\varphi \longmapsto \left[(x_0,\dots,x_n) \longmapsto \int_{\Gamma\backslash G} \varphi(\dot{g}x_0,\dots,\dot{g}x_n)\,d\mu(\dot{g})\right]$$



b/c $\varphi \in C_c((\mathbb{H}^n)^{n+1})^{\Gamma}$

$\varphi(\gamma\Delta) = \varphi(\Delta)$ but

$\varphi(g\Delta)$ not nec equal to $\varphi(\Delta)$

so we average over all g moving Δ w/in fundamental domain for $\Gamma \curvearrowright \mathbb{H}^n$.

<u>Fact</u>: $\tau \circ \rho^* = \mathrm{id}$. Since maps can only decrease norm,

Then $\|\omega_{\mathbb{H}^n}\| \geq \|\rho^*\omega_{\mathbb{H}^n}\| = \|\omega_M\| \geq \|\tau\,\omega_M\| = \|\omega_{\mathbb{H}^n}\|$

$\Rightarrow \quad \|\omega_M\| = \|\omega_{\mathbb{H}^n}\|.$