Tony Feng, Rational maps and blowups, 7-1-2016

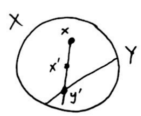
I. Rational maps

<u>Defn</u> $X \dashrightarrow Y$ <u>rational map</u> is actual map $U \longrightarrow Y$

where $U \subset X$ dense, Zariski open $\left(\begin{array}{l}\text{if } X \text{ connected} \\ U \text{ dense} \Leftrightarrow U \neq \emptyset\end{array}\right)$

So $U = X \setminus$ zeros of some polys

<u>Ex</u> $X = \mathbb{P}^n$ $\quad Y = \mathbb{P}^{n-1}$



$\mathrm{proj}_x : X \dashrightarrow Y$

$x' \longmapsto y'$

$U = X \setminus \{x\}$.

<u>Rmk</u> Change of coords so $x = [0, \dots, 0, 1]$ $\quad Y = (x_n = 0)$

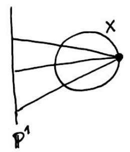
Then $\mathrm{proj}_x [x_0, \dots, x_n] = [x_0, \dots, x_{n-1}]$

which makes sense unless $x_0 = \dots = x_{n-1} = 0$ (ie unless pt is $[0 \dots 01]$)

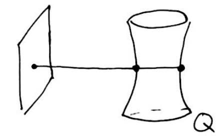
<u>Rmk</u> Can assume $X \setminus U$ has codim ≥ 2 $\left(\begin{array}{l}\text{this takes some alg geo work} \\ \text{show can extend rat'l map over} \\ \text{codim 1 sets}\end{array}\right)$

<u>Ex</u> $x \in X \subset \mathbb{P}^n$ restrict $\mathrm{proj}_x : X \dashrightarrow \mathbb{P}^{n-1}$.

(1) $X = \{A^2 + B^2 = C^2\} \subset \mathbb{P}^2$ $\rightsquigarrow$ iso plane conic to line.



(2) $Q \subset \mathbb{P}^3$ quadric (zero set of quadratic eqn)



$\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q \subset \mathbb{P}^3$

II. Blowing up

morphism = map defined everywhere

rational map = — " — on open subset.

<u>Ex</u> $x \in \mathbb{P}^n$ have $f_x : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

define $Bl_x \mathbb{P}^n = \overline{\text{Graph of } f_x \text{ in } \mathbb{P}^n \times \mathbb{P}^{n-1}}$

Have
$$\begin{array}{ccc} & Bl_x \mathbb{P}^n & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^n & \xrightarrow[f_x]{\quad\dashrightarrow\quad} & \mathbb{P}^{n-1} \end{array}$$

π_1 iso away from x

π_2 is map "resolving" f_x.

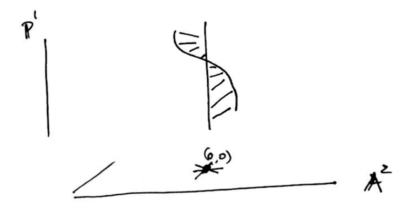
<u>Ex</u> $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$

$\cup$

$\mathbb{A}^2$

ψ

$(x, y) \longmapsto [x, y]$.



- $Bl_{(0,0)} \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ trivial fibration away from $(0,0)$
- fiber over $(0,0)$ is $\mathbb{P}^1$ = lines through $(0,0)$

General features

- $Bl_p X \longrightarrow X$ iso on $\pi^{-1}(X \setminus p)$.
- $\pi^{-1}(p) = \mathbb{P}(N_p X)$ $\quad N_p X$ normal bundle.

Blow up subvariety $Z \subset X$. cut out by $f_0, \ldots, f_n$.

$$X \overset{F}{\dashrightarrow} \mathbb{P}^n$$
$$p \longmapsto [f_0(p), \ldots, f_n(p)]$$

define $Bl_Z X = \overline{\text{graph of } F}$ in $X \times \mathbb{P}^n$

Prop If $F: X \dashrightarrow Y$ not defined on $Z \subset X$ $\exists$

$$\begin{tikzcd} & Bl_Z X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

Rmk $\pi^{-1}(Z) = \mathbb{P}(\cancel{N} N_Z X)$.

Sketch Pf Wlog $Y = \mathbb{P}^n$. $F = [f_0, \ldots, f_n]$ locally on X.

$$Z = (0 = f_0 = \cdots = f_n).$$

Now as in example
$$\begin{tikzcd} & Bl_Z X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed] & & \mathbb{P}^n \end{tikzcd}$$

and $\operatorname{Im}(\pi_2) \subset Y$ by topology. $\square$

Rmk Technically have to worry about $Z = (t^2 = 0)$ vs $Z = (t = 0)$.

~~For example~~ So need iterated blow up.

(If work w/ schemes This can be avoided...)

III. Zariski's Thm

Thm ~~Any~~ Any birational map of surfaces "obtained from blow ups"

Given $f: X \dashrightarrow Y$ $\exists$ $Z \subset X$ finite and seq of blow ups

$Bl_W Y \longrightarrow Y$ and iso $Bl_Z X \xrightarrow{\sim} Bl_W Y$.

Pf By previous discussion
$$\begin{tikzcd} & Bl_Z X \arrow[dl, "\pi"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

f birat $\Rightarrow$ $\exists$ $g: Y \dashrightarrow X$. Define $H = \pi^{-1} \circ g$

Lemma $H: Y \dashrightarrow Bl_Z X$ not defined at $q \in Y$ $\Rightarrow$ $h^{-1}(q) \overset{=C}{\text{ is}}$ a curve in X.

Having lemma $\Rightarrow$ can define $Bl_Z X \xrightarrow{G} Bl_W Y$

where W = Indet set of H:

for $q \in W$ $\quad h^{-1}(q) = C$ and define G on ~~...~~ $c \in C$

by taking normal direction at c and this gives normal dir

of $q \in Y$ which corresp to pt of $Bl_W Y$. $\square$