Local coefficients and Poincaré duality

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Poincaré duality is an important feature of manifolds that distinguishes them from other topological spaces. The simplest statement (the one given by Poincaré himself) says that for a closed, oriented, n-dimensional manifold M, there is an equality $b_k(M) = b_{n-k}(M)$ between the Betti numbers, where $b_i(M) = \dim H_i(M;\mathbb{Q})$. From a modern viewpoint, this equality follows from an isomorphism

$$H^i(M;\mathbb{Z}) \xrightarrow{\sim} H_{n-i}(M;\mathbb{Z})$$

induced by cap product with the fundamental class $[M] \in H_n(M;\mathbb{Z})$. The goal of this note is to state this theorem in a more general setting that applies to all compact manifolds, in particular manifolds with boundary and nonorientable manifolds. For the latter, one is lead to consider homology and cohomology with twisted coefficients.

1 Twisted coefficients

Let X be a finite, connected CW complex with fundamental group π. First we recall the definition of the *homology and cohomology groups with coefficients in* A, denoted $H_k(X;A)$ and $H^k(X;A)$ respectively, where A is any module over the group ring $\mathbb{Z}\pi$. Consider the cellular chain complex

$$\cdots \to C_2(\tilde{X}) \xrightarrow{\partial} C_1(\tilde{X}) \xrightarrow{\partial} C_0(\tilde{X}) \to 0$$

of the universal cover $\tilde{X}$. The deck group action on $\tilde{X}$ makes $C_k(\tilde{X})$ a $\mathbb{Z}\pi$-module. Given another $\mathbb{Z}\pi$-module A, one obtains further chain complexes from the abelian groups $C_k(\tilde{X}) \otimes_{\mathbb{Z}\pi} A$ and $\operatorname{Hom}_{\mathbb{Z}\pi}\big(C_k(\tilde{X}), A\big)$. The groups $H_*(X;A)$ and $H^*(X;A)$ are defined as the homology groups of these chain complexes, respectively.

Remark (left vs. right modules)**.** We briefly remark on a technical point in the above definition. For a ring R and for R-modules A and B, the definitions of $A \otimes_R B$ and $\operatorname{Hom}_R(A,B)$ require some care in specifying A and B as left or right modules. For $A \otimes_R B$, one requires A to be a right module and B to be a left module, while for $\operatorname{Hom}_R(A,B)$, both A and B should have the same handedness (see [1, §5.1]). Nevertheless, when $R = \mathbb{Z}G$ is a group ring, $A \otimes_R B$ and $\operatorname{Hom}_R(A,B)$ are always defined, since any right module can be viewed as a left module[1] and vice versa.

Remark (relation to ordinary cohomology)**.** Twisted coefficients can be used to recover the homology of X and any of its covers:

1. To recover $H_k(X;\mathbb{Z})$ and $H^k(X;\mathbb{Z})$ (defined as the (co)homology of the cellular chain complex of X), use the trivial module $A = \mathbb{Z}$, and note there are isomorphisms

$$C_k(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{Z} \simeq C_k(X) \quad \text{and} \quad \operatorname{Hom}_{\mathbb{Z}\pi}\big(C_k(\tilde{X}), \mathbb{Z}\big) \simeq \operatorname{Hom}\big(C_k(X), \mathbb{Z}\big).$$

[1] A right ZG-module A is a left module via action $g.a := a.g^{-1}$ for $a \in A$ and $g \in G$.

2. The homology $H_k(\tilde{X};\mathbb{Z})$ is isomorphic to $H_k(X;\mathbb{Z}\pi)$, since $C_k(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi = C_k(\tilde{X})$. In a similar way, for any cover $\hat{X} \to X$, there is an isomorphism

$$H_k(\hat{X};\mathbb{Z}) \simeq H_k(X;\mathbb{Z}\pi \otimes_{\mathbb{Z}\hat{\pi}} \mathbb{Z}) \tag{1}$$

where $\hat{\pi} = \pi_1(\hat{X})$. Here the coefficient module $\mathbb{Z}\pi \otimes_{\mathbb{Z}\hat{\pi}} \mathbb{Z}$ is isomorphic to $\mathbb{Z}\{\pi/\hat{\pi}\}$, the free abelian group on the set $\pi/\hat{\pi}$.

When one repeats this discussion for cohomology, one finds that $H^k(X;\mathbb{Z}\pi) \simeq H^k_c(\tilde{X};\mathbb{Z})$ (compactly supported cohomology), since a $\mathbb{Z}\pi$-equivariant cochain $C_k(\tilde{X}) \to \mathbb{Z}\pi$ is equivalent to a cochain $C_k(\tilde{X}) \to \mathbb{Z}$ supported on finitely many simplices.

Example (the orientation module)**.** Any manifold M admits an *orientation double cover* $\hat{M} \to M$, which is determined by a homomorphism $w : \pi_1(M) \to \mathbb{Z}/2 \simeq \mathrm{Aut}(\mathbb{Z})$ (the first Stiefel–Whitney class). The induced $\mathbb{Z}\pi$-module is called the *orientation module* and denoted $\mathbb{Z}_w$.

Example (computation for $\mathbb{R}P^2$)**.** Let's compute $H_*(M;\mathbb{Z}_w)$, where $M = \mathbb{R}P^2$ and $\mathbb{Z}_w$ is the orientation module of the preceding example. Give M the cell structure that has a single cell c_i in each dimension $0 \le i \le 2$. The cellular chain complex of the universal cover $\tilde{M} \simeq S^2$ has the form

$$0 \to \mathbb{Z}\pi \xrightarrow{\partial_2} \mathbb{Z}\pi \xrightarrow{\partial_1} \mathbb{Z}\pi \to 0, \tag{2}$$

where $\mathbb{Z}\pi \simeq \mathbb{Z}[t]/(t^2-1)$ is the group ring of $\pi_1(M) = \mathbb{Z}/2$. The differential ∂_2 is multiplication by $1+t$, and ∂_1 is multiplication by $1-t$. The orientation module $\mathbb{Z}_w$ is isomorphic to the submodule $\mathbb{Z}^- < \mathbb{Z}\pi$ generated by $1-t$. It follows that $C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}_w$ is isomorphic to a subcomplex of (2), and the homology $H_*(\mathbb{R}P^2;\mathbb{Z}_w)$ is computed from the chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0.$$

This gives $H_2(\mathbb{R}P^2;\mathbb{Z}_w) = \mathbb{Z}$ and $H_1(\mathbb{R}P^2;\mathbb{Z}_w) = 0$ and $H_0(\mathbb{R}P^2;\mathbb{Z}_w) = \mathbb{Z}/2$.

2 Poincaré duality

Throughout this section M will be a compact, connected n-manifold with fundamental group π. We begin by explaining the statement of Poincaré duality in the case when M is closed and oriented. Then we explain what's different in the case M has nonempty boundary or is nonorientable. Our aim here is to be brief and to give a flavor for what is involved; for more details, we refer the reader to [1, Chapters 3,5] and [3, Chapter 4].

2.1 Duality for closed oriented manifolds

Cap products. For $z \in H_j(M;\mathbb{Z})$, the cap product is a homomorphism

$$\begin{array}{rccc} z\frown & : & H^k(M;\mathbb{Z}) & \to & H_{j-k}(M;\mathbb{Z}) \\ & & \phi & \mapsto & z \frown \phi, \end{array}$$

defined for $k \le j$. In fact this homomorphism exists with $\mathbb{Z}$ replaced by any $\mathbb{Z}\pi$-module A. Cap products are dual to cup products in the sense that

$$\langle \phi \smile \psi, z\rangle = \langle \psi, z \frown \phi\rangle$$

for $\phi \in H^k(M)$, $\psi \in H^{j-k}(M)$, and $z \in H_j(M)$. Similar to cup products, cap products can be defined on the chain level by choosing a chain approximation for the diagonal, as is nicely explained in [1, Chapter 3]. For explicit formulas, see [2, §3.3]. These formulas won't play a role in the discussion that follows.

Definition. An orientation on a closed n-manifold determines a generator $[M] \in H_n(M;\mathbb{Z}) \simeq \mathbb{Z}$, which is called the *fundamental class.*

Theorem 2.1 (Poincaré duality)**.** *Let M be a closed, oriented n-manifold with fundamental class $[M] \in H_n(M;\mathbb{Z})$. For each $\mathbb{Z}\pi$-module A and for $0 \leq k \leq n$, the cap product $[M] \frown : H^k(M;A) \to H_{n-k}(M;A)$ is an isomorphism.*

2.2 Duality for all compact manifolds

For a compact, orientable manifold M with boundary ∂M the cap product involves relative (co)homology: for $z \in H_j(M, \partial M;\mathbb{Z})$ and for A a $\mathbb{Z}\pi$-module, then there are homomorphisms

$$\begin{aligned} z \frown : &\quad H^k(M;A) \to H_{j-k}(M,\partial M;A) \\ z \frown : &\quad H^k(M,\partial M;A) \to H_{j-k}(M;A). \end{aligned}$$

The version of Poincaré duality for manifolds with boundary is frequently called Lefschetz duality.

Theorem 2.2 (Lefschetz duality)**.** *Let M be a compact, oriented n-manifold with boundary ∂M and with fundamental class $[M] \in H_n(M,\partial M;\mathbb{Z})$. For each $\mathbb{Z}\pi$-module A and for $0 \leq k \leq n$, the cap product $[M] \frown$ defines isomorphisms*

$$H^k(M;A) \simeq H_{n-k}(M,\partial M;A) \quad \text{and} \quad H^k(M,\partial M;A) \simeq H_{n-k}(M;A).$$

Now we discuss the case where M is closed and nonorientable. Our main task is to define the fundamental class in $H_n(M;\mathbb{Z}_w)$, where $\mathbb{Z}_w$ is the orientation module. The definition will follow from the next proposition.

Proposition 2.3. *Let M be a closed n-manifold with orientation cover $\kappa : \hat{M} \to M$. There is an exact sequence*

$$0 \to H_n(M;\mathbb{Z}_w) \to H_n(\hat{M};\mathbb{Z}) \xrightarrow{\kappa_*} H_n(M;\mathbb{Z}).$$

To define the fundamental class, fix an orientation on $\hat{M}$. If M is orientable, we assume that the orientation on $\hat{M} \simeq M \sqcup M$ is such that the two components have opposite orientations. In the nonorientable case $H_n(M;\mathbb{Z}) = 0$ [2, Theorem 3.26], so in either case, the fundamental class $[\hat{M}] \in H_n(\hat{M};\mathbb{Z})$ is in $\ker(\kappa_*)$.

Definition. The *fundamental class* $[M]$ *of* M is defined as the class $[\hat{M}] \in H_n(M;\mathbb{Z}_w) = \ker\big[H_n(\hat{M};\mathbb{Z}) \to H_n(M;\mathbb{Z})\big]$.

Compare with [3, Definition 4.50]. When M is orientable, this agrees with the previous definition. The proof of Proposition 2.3 is a nice application of the discussion in Section 1.

Proof of Proposition 2.3. Let $\pi = \pi_1(M)$ and $\hat{\pi} = \pi_1(\hat{M})$. Consider the short exact sequence

$$0 \to \mathbb{Z}^- \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z} \to 0 \tag{3}$$

of $\mathbb{Z}[\mathbb{Z}/2]$-modules, where $\mathbb{Z}^-$ is the (unique) nontrivial simple module over $\mathbb{Z}[\mathbb{Z}/2]$. The sequence (3) is also an exact sequence of $\mathbb{Z}\pi$ modules since κ defines a quotient $\pi \to \mathbb{Z}/2$. Then (3) induces a short exact sequence of chain complexes

$$0 \to C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}_w \to C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}[\mathbb{Z}/2] \to C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{Z} \to 0.$$

The induced long exact sequence on homology contains the exact sequence claimed in the proposition. □

Theorem 2.4 (Duality for nonorientable manifolds)**.** *Let M be a closed n-manifold with orientation module $\mathbb{Z}_w$ and with fundamental class $[M] \in H_n(M;\mathbb{Z}_w)$. For each $\mathbb{Z}\pi$-module A and for $0 \leq k \leq n$, the cap product $[M]\frown : H^k(M;A) \to H_{n-k}(M;A \otimes \mathbb{Z}_w)$ is an isomorphism. In particular, we have isomorphisms $H^k(M;\mathbb{Z}) \simeq H_{n-k}(M;\mathbb{Z}_w)$.*

Exercise. Check that the theorem agrees with our computation of $H_*(\mathbb{R}P^2;\mathbb{Z}_w)$ in Section 1.

Combining Theorems 2.2 and 2.4, we arrive at the following statement of Poincaré duality that applies to all compact manifolds.

Theorem 2.5 (Duality for compact manifolds)**.** *Let M be a compact n-manifold with orientation module $\mathbb{Z}_w$, boundary ∂M (possibly empty), and fundamental class $[M] \in H_n(M, \partial M;\mathbb{Z}_w)$. For each $\mathbb{Z}\pi$-module A and for $0 \leq k \leq n$, the cap product $[M]\frown$ defines isomorphisms*

$$H^k(M;A) \simeq H_{n-k}(M,\partial M;A \otimes \mathbb{Z}_w) \quad \textit{and} \quad H^k(M,\partial M;A) \simeq H_{n-k}(M;A \otimes \mathbb{Z}_w).$$

References

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