Bulgarian Solitaire in Three Dimensions

Anton Grensjö
antongrensjo@gmail.com

under the direction of
Henrik Eriksson
School of Computer Science and Communication
Royal Institute of Technology

Research Academy for Young Scientists
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Abstract

Bulgarian Solitaire is a mathematical game played in the universe of integer partitions. It can be represented as having a number of items divided into separate piles. The operation of the game consists of taking one item from each pile and creating a new pile from the collected items. This results in a new configuration of the items. The operation is then applied over and over again. In this study we discuss and prove several properties of this game, such as convergence and cycle lengths. The proofs are based on observations of the behavior of the game, and are illustrated using rotated Young diagrams. The main purpose is however to define a new, three-dimensional (3D) version of the game, and explore its properties. This is done by defining the game on plane partitions, which can be visualized using three-dimensional Young diagrams. In the 3D version we define six different moves, each based on executing the original operation on different layers of the Young diagram individually.

1 Introduction

The known history of Bulgarian Solitaire began around 1980, when Konstantin Oskolkov of the Steklov Mathematical Institute in Moscow met a man on a train who introduced him to a simple game, which can be described as follows: imagine fifteen playing cards in front of you. Arrange these cards into a number of piles. For consistency, keep the piles sorted in order of decreasing height. Now, pick one card from each pile and create a new pile from these cards. Repeat this step over and over again. Here are the pile heights from an example execution:

$$(7,4,2,2) \Rightarrow (6,4,3,1,1) \Rightarrow (5,5,3,2) \Rightarrow (4,4,4,2,1) \Rightarrow (5,3,3,3,1)$$
$$\Rightarrow (5,4,2,2,2) \Rightarrow (5,4,3,1,1,1) \Rightarrow (6,4,3,2) \Rightarrow (5,4,3,2,1) \Rightarrow (5,4,3,2,1)$$

Note that the final state is stable - it leads back to itself. When Oskolov reached this configuration he became intrigued and tried again a few times with different initial configurations. Every time the result turned out to be the same: $(5,4,3,2,1)$. Oskolov told his colleagues about the game and it started to spread [1]. Via Bulgaria it reached Stockholm and Henrik Eriksson, who gave the game its name when he published an article about it in 1981 [2]. It then spread all over the world, referred to as *Bulgarian Solitaire*.

At first mathematicians were stumped, but it was soon proved that when the number of cards is a triangular number (a number in the form $1+2+3+...+N$) the game always converges to the state $(N, N-1, ..., 2, 1)$, which has a cycle of length 1, regardless of the initial distribution [3]. When the number of cards is not a triangular number the game converges to a longer cycle. Many questions arose, such as how many cycles there are for an arbitrary number of cards, the lengths of these cycles and the largest possible number of moves that can be performed before the game reaches a cycle. Although these questions have already been answered, some areas are still unexplored.

This study introduces a version of Bulgarian Solitaire extended to three dimensions, and investigates the properties of that version. Furthermore, some simple new proofs for properties of the original game are presented.

2 Bulgarian Solitaire

2.1 Integer partitions

Bulgarian Solitaire can be described formally by making the abstraction from piles of playing cards to integer partitions. A *partition of* n is a t-tuple λ of integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{t-1}, \lambda_t) \tag{1}$$

whose sum is n ($\sum_{i=1}^{t} \lambda_i = n$). The integers $\lambda_1, \lambda_2, ..., \lambda_t$ are called *parts* of the partition. To express that a partition λ is a partition of n, we write $\lambda \vdash n$. We also denote the number of parts $|\lambda| = t$. The order of the parts in a partition is usually not of significance, but for practical reasons we choose to order and index them in non-increasing order:

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_{t-1} \geq \lambda_t \tag{2}$$

For example, $\lambda = (3, 2, 1) \vdash 6$, since $3 + 2 + 1 = 6$.

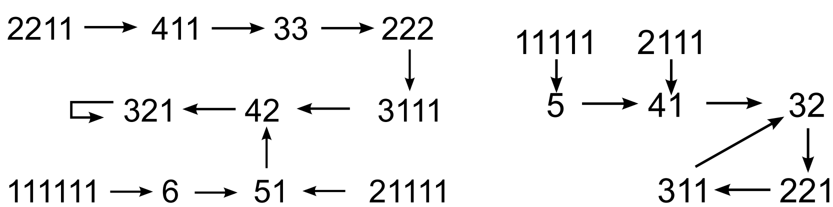
Bulgarian Solitaire is now defined as an operation $B(\lambda)$ on partitions,

$$B(\lambda) = (|\lambda|, \lambda_1 - 1, \lambda_2 - 1, ..., \lambda_t - 1) \tag{3}$$

where parts of size 0 are discarded if they exist and where the parts are reordered to be consistent with equation (2). This definition is equivalent to the more informal one in section 1: each part λ_i of a partition corresponds to a pile of cards, and the sizes of the parts correspond to the heights of the piles.

2.2 The game graph

All information about how Bulgarian Solitaire operates on partitions of an integer n can be contained in a directed graph, which we call the *game graph of* n. The graph consists of all possible partitions of n and their relations, visualized as arrows where each arrow points towards the partition which comes next in the game (directed edges). There are two examples in figure 1.



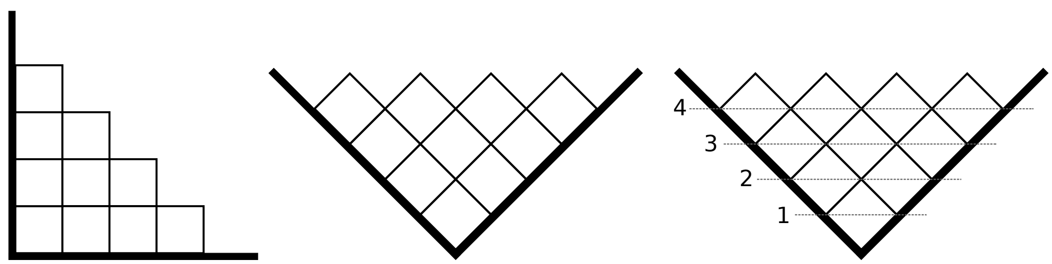
(a) Game graph of 6. (b) Game graph of 5.

Figure 1: The game graphs of 6 and 5.

In figure 1a we see that for 6, which is a triangular number, all partitions lead to the position $(3, 2, 1)$. For 5, which is not a triangular number, all partitions instead lead to a single cycle of length 3 (see figure 1b). The main questions regarding Bulgarian Solitaire are about properties of the game graph. This paper will discuss the lengths of the cycles and characterize the Garden of Eden partitions, that is: the partitions which are impossible to achieve, unless one begins there.

2.3 Visual representation

Partitions are often graphically represented by Young diagrams (see figure 2a). The columns of a Young diagram each correspond to a part λ_i of the partition and the height of a column shows the size of the same part. Young diagrams are usually drawn as in figure 2a, but the mathematician Anders Björner came up with the idea to rotate the diagram 45° counter-clockwise[1], in order to obtain a more intuitive way to illustrate Bulgarian Solitaire (figure 2b). We will see that the squares will move consistently with how they would have moved if they were affected by gravity.



(a) Standard form (b) Rotated diagram (c) Enumeration of diagonals

Figure 2: Young diagrams

We define *diagonals* in a rotated Young diagram as in figure 2c. Also, the *length* of a diagonal is the number of positions on it (which is the same as its index). Now we observe what happens when performing the Bulgarian Solitaire operation B. Take a look at figure 3. First, between step (1) and (2), the rightmost row is removed, which corresponds to taking one card from each pile, and the remaining squares fall down a step. Then, between step (2) and (3) the removed row is rotated 90° and inserted again, which corresponds to creating a new pile of cards. Note that the dark square remains in the same diagonal after this operation, it just moves one step to the right. If the procedure is repeated the same thing will happen, but the square will instead move to the leftmost position on its diagonal (it is a cyclic permutation).

[1] According to personal communication with Henrik Eriksson.

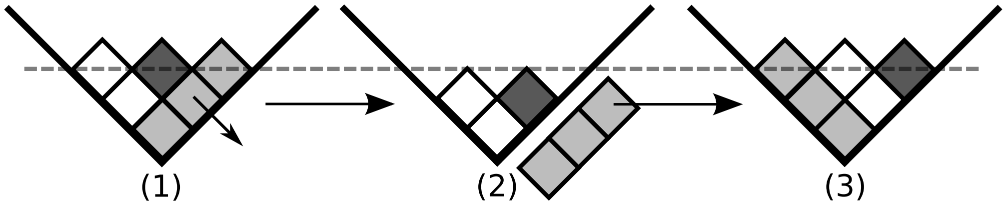


Figure 3: Bulgarian Solitaire on the partition $\lambda = (3, 2, 1)$.

Figure 4 shows a slightly different example. The same procedure is performed, but this time the inserted column is shorter than the one already there. This means that it will be inserted in an incorrect position, as in step (3), where the columns are not sorted in non-increasing order. Although, if one imagines the diagram being affected by gravity, one understands that the dark square should slide down, which leaves the columns sorted in correct order as in step (4). Notice also that the dark square moves to a different diagonal during this procedure; it fills a hole in the diagonal beneath it. We say that in (1), the dark square was in a *non-optimal diagonal*, since there was a hole in the diagonal beneath. In (4) it is in an optimal diagonal, since there are no holes it can fill.

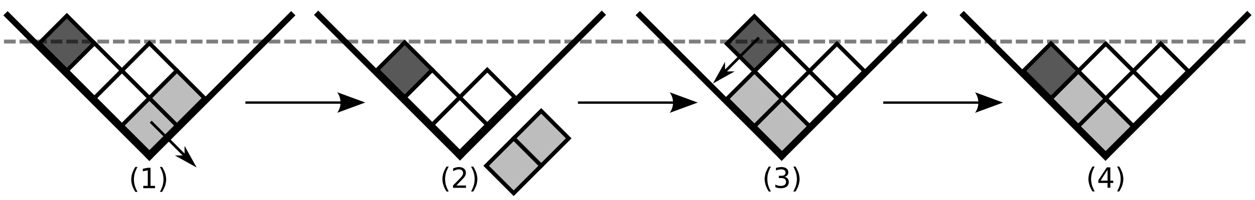


Figure 4: Bulgarian Solitaire on a rotated Young diagram. (1) is the partition $\lambda = (4, 2)$ and (4) is the partition $B(\lambda) = (3, 2, 1)$.

2.4 Convergence for triangular numbers

Definition 1. *The k^{th} triangular number T_k is defined as follows:*

$$T_k = 1 + 2 + 3 + ... + k$$

Lemma 1. *When playing Bulgarian Solitaire, if, in the Young diagram of the current partition, there is at least one square in a non-optimal diagonal (a diagonal for which there is at least one empty place in the diagonal below), a square will eventually drop down to the diagonal below.*

Proof. Consider an empty position in the rotated Young diagram of λ which is not in the topmost diagonal (from now on referred to as a *hole*) and a square in the diagonal above. Let the hole be in the k^{th} diagonal and the square in the $(k+1)^{th}$ diagonal. Notice that the square and the hole will cycle on their respective diagonals when applying the operation $B(\lambda)$ multiple times. Because the hole has cycle length k and the square has cycle length $k+1$ they will shift one step with respect to each other every n moves (see figure 5). This implies that the square will eventually be placed on top of the hole, and consequently fall down into it, as in figure 4 (3). If, during this procedure, the square should fall down into another hole or the hole should be filled by another square, the lemma is still fulfilled. □

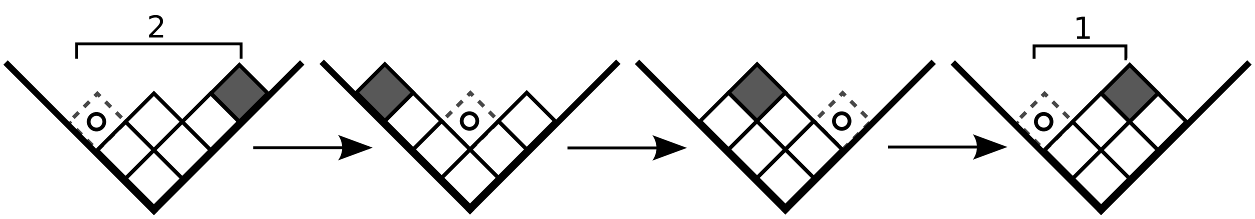


Figure 5: An example of how a hole and a square move relative to each other. Note that the distance between them decreases by one every three steps (in this particular case).

Theorem 1. *When performing Bulgarian Solitaire on an initial partition $\lambda \vdash T_k$ (a partition of a triangular number) one will always reach the state $(k, k-1, ..., 2, 1)$ after a finite number of operations.*

Proof. By lemma 1, all squares in non-optimal diagonals will eventually drop down to lower diagonals. Since T_k squares fill exactly k diagonals, the only partition where no square is in a non-optimal diagonal is the stable partition, where all diagonals are filled, that is:

$(k, k-1, ..., 2, 1)$. □

Corollary 1. *The only cycle length that exists in the game graphs of triangular numbers is the length 1.*

Proof. By theorem 2.4 one will always reach the state $(k, k-1, ..., 2, 1)$ when starting with a partition $\lambda \vdash T_k$. By equation (3) in section 2.1, if $\lambda = (k, k-1, ..., 2, 1)$, then $B(\lambda) = (k, k-1, k-2, ..., 1)$. □

2.5 Cycle lengths

As previously mentioned, Bulgarian Solitaire always converges for partitions of T_k, but it is easy to realize that all games of Bulgarian Solitaire eventually must return to an already visited state, since there are only a finite number of partitions of n. However, there is no partition of a non-triangular number which leads immediately back to itself, since there will always be a few extra squares circulating on the topmost diagonal (by lemma 1 all holes will be filled). Instead, the game will converge to a cycle of length longer than 1. There might be multiple different cycles in a game graph, and the question is how long these cycles are.

Theorem 2. *Let G be the game graph of $T_k + r$, where $T_k < T_k + r < T_{k+1}$. Then, for every common divisor d of both $k+1$ and r there is a cycle of length $\frac{k+1}{d}$ in G, and there are no cycle lengths not fulfilling this criteria.*

Proof. When all holes in a partition have been filled, we have r squares which are circulating on the $(k+1)^{th}$ diagonal, which has length $k+1$. Thus, we can think of a position as the distribution of r items over $k+1$ positions on a circle. When applying the operation B, the squares will get rotated one step around the circle (see figure 6).

For all divisors d of both $k+1$ and r we can construct a cycle as follows. Divide the r squares

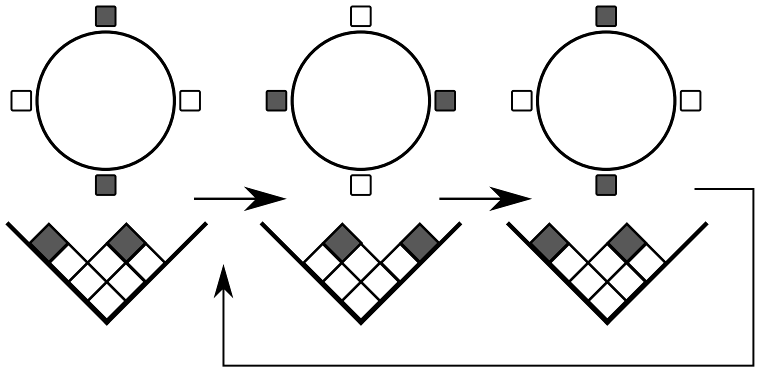


Figure 6: Here is an example of a cycle with length 2, represented both as rotated Young diagrams and as items around a circle. The leftmost and rightmost positions are identical. For this cycle: $k+1=4$, $r=2$, $d=2$

into d identical groups of r/d elements each. Now place these groups symmetrically over the circle. This is possible since d is a factor of $k+1$, and we will get an offset of $l=\frac{k+1}{d}$ between each group. That is, after l operations, the game will be in a state identical to the first one. Thus, we have the cycle length $l=\frac{k+1}{d}$.

But are these cycle lengths the only possible lengths? If l is one of the lengths described, then l is a factor of $k+1$ and $d=\frac{k+1}{l}$ is a factor of r. We shall show that both of these conditions are necessary. Firstly, $k+1$ must be a multiple of the cycle length, since we can always get to a state identical to the first one in $k+1$ steps. Secondly, in order to get a cycle length $l \leq k+1$ it has to be possible to divide the squares into $d=\frac{k+1}{l}$ identical groups. Therefore d must be a factor of the number of circulating squares, r. $\square$

2.6 Garden of Eden

Garden of Eden partitions (named by a biblical analogy) are partitions which can not be reached unless one begins the game there, and are impossible to get back to, once one has left. Graph theoretically a Garden of Eden partition is a node in the game graph with in-degree 0.

Theorem 3. *A partition λ is a Garden of Eden partition if and only if $\lambda_1 < |\lambda| - 1$, that is: if the highest column of the Young diagram has fewer squares than the remaining number of columns.*

Proof. Suppose that $\lambda_1 < |\lambda| - 1$ and that there is a predecessor τ to λ. When performing the operator B on τ, a new part is created, of size $|\tau|$. Since $|\tau| \geq |\lambda| - 1$ (if all parts are still nonzero after applying B, $|\lambda| = |\tau| + 1$), the size of this part is *at least* $|\lambda| - 1$. But the largest part of λ was by the assumption smaller than this. Since this is a contradiction, the partition λ cannot possibly have a predecessor and is therefore a Garden of Eden partition.

Now suppose that $\lambda_1 \geq |\lambda| - 1$. Then we can construct a predecessor as follows: remove the biggest part of λ, λ_1, and add 1 to each remaining part (this is possible thanks to the assumption). Now, if $\lambda_1 > |\lambda| - 1$, add $\lambda_1 - (|\lambda| - 1)$ new parts of size 1. When performing the operator B on this new partition, we will get a new part of size λ_1, all parts of size 1 are removed and the remaining parts are decreased by one. This leaves us with the original partition λ. □

2.7 The dual game

Another suggested way to look at Bulgarian Solitaire is the *dual game*. While the original game is based on the principle of taking one card from each pile and forming a new pile, the dual game does the opposite: it takes the biggest pile and hands out the cards from that pile to the remaining piles. This corresponds to taking the leftmost column of the Young diagram, rotating it 90° clockwise and inserting it as the bottom row (see figure 7). This can be described using conjugates of partitions.

Definition 2. *The conjugate λ' of a partition λ is the partition that corresponds to the Young diagram obtained by mirroring the Young diagram of λ in the sense that rows are turned into*

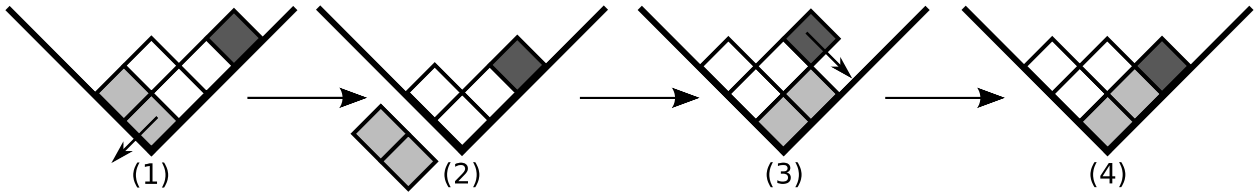


Figure 7: The dual game on the partition $\lambda = (2, 2, 1, 1)$. Notice that the exact same thing happens if we mirror the diagram horizontally, perform regular Bulgarian Solitaire on it, and then mirror it back (compare to figure 4).

columns and columns are turned into rows.

Definition 3. *The operator of the dual game, B', is defined as follows:*

$$B'(\lambda) = (B(\lambda'))' \tag{4}$$

That is, the dual game can be performed by mirroring the partition (taking its conjugate), performing the original operator B on the mirrored partition, and finally mirroring it back. Note that this is equivalent to creating a new row from the leftmost column in the Young diagram, instead of doing the usual procedure of creating a new column from the bottom row.

Theorem 4. *All previously stated properties of Bulgarian Solitaire also apply to the dual game (except for those involving right or left; the dual game is mirrored), i.e. the two games are isomorph.*

Proof. Since the dual game can be seen as Bulgarian Solitaire on the conjugates of the partitions, the exact same properties must apply, except for the fact that the game is mirrored. □

Also, from the visual interpretation of the dual game follows that B and B' cancel each other out, given that no holes are filled when applying either of the operations. B makes the squares of the Young diagram rotate to the right on their diagonals and B' makes them rotate to the left. All these properties of the dual game will be useful when defining Bulgarian Solitaire in three dimensions.

2.8 Possible applications

Bulgarian Solitaire may seem to be a completely abstract game, lacking practical applications. To a certain extent that might be true, but there are acually connections between Bulgarian Solitaire and real world phenomena. One example is taxes. The government collects a small amount of money from each citizen, which is put into the public treasury. This corresponds to regular Bulgarian Solitaire; each part of the partition is decreased by a small amount, and the sum of these decreases forms a new part. Generally though, the government takes a predefined percentage of the income of each citizen, instead of a constant amount, but one might also define and explore the properties of such a variant of Bulgarian Solitaire. Generally, every phenomena where something is taken from many entities and collected to another entity could have common properties with Bulgarian Solitaire. The dual game, on the other hand, applies the principles of Robin Hood: taking from the rich and giving it to the poor. Lemma 1 actually implies that if Robin Hood continues to take from the rich and give to the poor, the distribution of fortune in his community will become more triangular (like the shape of the stable state) over time. Reorganization of companies is another example. The board of directors might choose to remove one department and distribute the employees from that department evenly over the remaining departments, which corresponds to the dual game.

3 Extension into three dimensions

3.1 Compatibility with the partition lattice

We define Young's lattice (seen in figure 8) as a partially ordered set which describes inclusion of Young diagrams. In other words, the lattice contains information for every pair of Young diagrams whether it would be physically possible to place one Young diagram three-dimensionally on top of the other, without any squares falling down due to gravity.

Definition 4. *A Young diagram of a partition* $\tau = (\tau_1, \tau_2, ..., \tau_t)$ *is included in the Young diagram of a partition* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)$ *if and only if* $t \leq s \ \wedge \ \forall i, 1 \leq i \leq t \ : \ \tau_i \leq \lambda_i$.

Inclusion of τ in λ is from now on denoted by $\tau \leq \lambda$ or $\lambda \geq \tau$.

In the visual representation of Young's lattice there is an arrow from one partition to another of adjacent size if the first one is included in the second one. Since the inclusion relation is transitive, a Young diagram in the lattice is included in another if there is a directed path from the first one to the second one. Notice also that the diagram is arranged into horizontal levels, where the n^{th} level consists of all partitions of n.

Theorem 5. *Bulgarian Solitaire is compatible with the lattice order, i.e:* $\tau \leq \lambda \ \Rightarrow \ B(\tau) \leq B(\lambda)$. *That is: If a Young diagram includes another Young diagram, that will still be the case after applying the Bulgarian Solitaire operation on both of them.*

Proof. Consider the Young diagram of a partition τ placed three-dimensionally on top of the diagram of a partition λ, for which $\tau \leq \lambda$. Since λ includes τ, every square in the diagram of τ is placed on top of a square in λ. Consider a pair of squares lying on top of each other. Now, perform the operation B on both partitions simultaneously. In the most common case both of the squares move one position within their diagonals, and thereby stay on top of each other. If both of them should fall into a hole in the diagonal below they also stay on

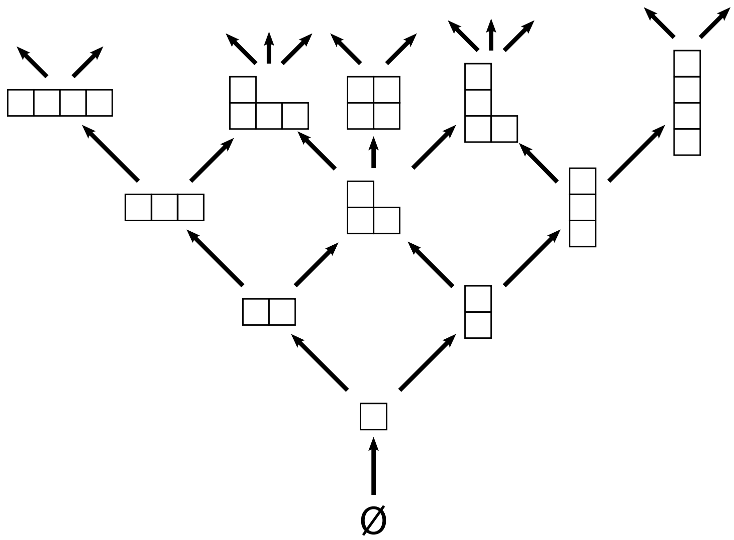


Figure 8: The Hasse diagram of Young's lattice drawn for $n \leq 4$.

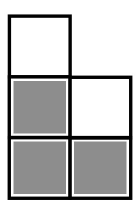


Figure 9: The Young diagram of $\tau = (2, 1)$ is included in the Young diagram of $\lambda = (3, 2)$.

top of each other. The last case is if the square from τ falls into a hole but the square from λ does not. Then there has to be another square from λ in the new position of the square from τ, since otherwise the square from λ would also have fallen. The case that the square from λ falls but not the square from τ will never arise, since that contradicts the initial assumption that $\tau \leq \lambda$. Therefore, if a square from τ lies on top of a square from λ before B is applied, it still does afterwards. This applies to all squares in τ. □

Corollary 2. *If an arbitrary number of partitions form a chain in Young's lattice (where each chosen partition is included in all subsequently chosen partitions), then after performing the Bulgarian Solitaire operation B on all of them, they still form a chain in Young's lattice and are still compatible in the same way.*

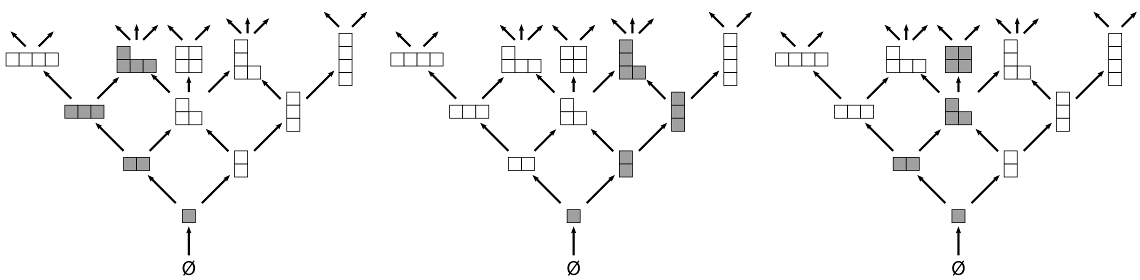


Figure 10: This is an example of 4 parallel games. The states of the respective games are marked with gray in Young's lattice. Between each figure the operation B has been applied on all 4 partitions. Notice that the currently chosen partitions stay under each other in the lattice during all steps, which conforms with corollary 2

3.2 Plane partitions

In order to represent Bulgarian Solitaire on the partition lattice, we introduce *plane partitions*. Plane partitions of n are like usual integer partitions in the sense that the sum of their parts is n, but instead of just being a list of integers a plane partition forms a two-dimensional grid of integers [4].

Definition 5. *Define a* plane partition *of n as an array $\pi = (\pi_{ij})$, where $i, j \geq 1$, all π_{ij}*

are nonnegative integers and $\sum \pi_{ij} = n$*. Every row and column should be sorted in non-increasing order, that is:*

$$\forall i, j : \pi_{i,j} \geq \pi_{i+1,j} \ \wedge \ \pi_{i,j} \geq \pi_{i,j+1} \tag{5}$$

A plane partition can be represented by a three-dimensional Young diagram (see figure 11). Here, each horizontal layer can be interpreted as a partition from Young's lattice. We denote the horizontal layers with $b_1, b_2, b_3, ...$ where the lowest layer has index one, the second has index two, and so on. Note that the Young diagram of a layer b_i includes all layers above it. Therefore a plane partition can be interpreted as a chain in Young's lattice.

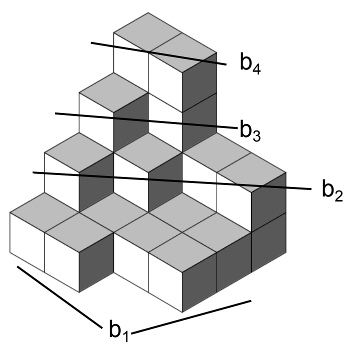


Figure 11: A visualization of the following plane partition of 26:

$$\begin{array}{cccc} 4 & 4 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & & \end{array}$$

Notice that the 3D diagram is aligned and sorted against three sides: the left side (denoted l), the right side (denoted r) and the bottom (denoted b). Then, by symmetry, it follows that the layers parallel with the left side (denoted $l_1, l_2, ...$) and the layers parallel with the right side (denoted $r_1, r_2, ...$) also form chains in Young's lattice. Generally:

$$s_1 \geq s_2 \geq s_3 \geq ... \tag{6}$$

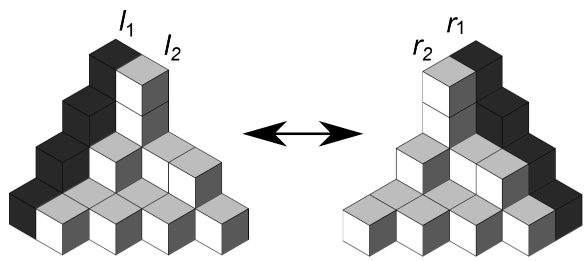


Figure 12: The operations $B'(\pi, b)$ and $B(\pi, b)$ visualized. When performing operations with respect to b, the left layer rotates 90° clockwise and is inserted on the right, or the right layer is rotated counter-clockwise and inserted on the right.

3.3 The 3D game

Definition 6. *We define Bulgarian Solitaire on plane partitions with two basic operations, $B(\pi, s)$ and $B'(\pi, s)$, where π is a plane partition and s is one of the sides b, l and r with respect to which the operation will be performed.*

$B(\pi, s)$ performs the original Bulgarian Solitaire operator B on the partitions of all layers parallel to the side s, that is: $s_1, s_2, \ldots$.

$B'(\pi, s)$ performs the dual game operator B' on the partitions of all layers parallel to the side s, that is: $s_1, s_2, \ldots$.

This definition is possible thanks to the results of corollary 2, since applying $B(\pi, s)$ or $B'(\pi, s)$ can be seen as applying $B(\lambda)$ or $B'(\lambda)$ on a chain of diagrams in Young's lattice. The conservation of the lattice order (see theorem 5 and corollary 2) is equivalent to the plane partition still being arranged in a way consistent with equation (5).

The definition of Bulgarian Solitaire in 3D opens up for six different possible moves: $B(\pi, b)$, $B'(\pi, b)$, $B(\pi, l)$, $B'(\pi, l)$, $B(\pi, r)$ and $B'(\pi, r)$. This makes the game more complex, but also changes the nature of the game: it is no longer deterministic, the player can now choose between moves.

3.4 Convergence

For two-dimensional Bulgarian Solitaire we have shown that squares in a Young diagram either stays in the same diagonal or drops to a lower one when B or B' is applied (see figure 3 and 4). The same thing applies to the 3D game. A cube in a three-dimensional Young diagram is able to traverse the diagonals of the layers it occupies. For example, if $B(\pi, b)$ is applied, all cubes will cycle on diagonals parallel to the bottom layer. The set of positions to which a cube can be moved (assuming it does not fall into a hole in the process) using the six operations form a diagonal plane. For exemple, all visible cubes in figure 13 form a diagonal plane. A cube can never move to a higher diagonal plane; it can either stay in its current plane or fall into a hole in a lower plane.

Definition 7. *The tetrahedral number P_k is the sum of the k first triangular numbers. That is:*

$$P_k = \sum_{i=1}^{k} T_i = \sum_{i=1}^{k} \sum_{j=1}^{i} j \tag{7}$$

The tetrahedral number P_k can also be expressed with the following formula (easily provable by induction):

$$P_k = \frac{n(n+1)(n+2)}{6} \tag{8}$$

There is a three-dimensional equivalence to the two-dimensional stable state, as visualized in figure 13. This state loops back to itself, no matter which of the six different moves is performed. The stable form exists in the 3D game graphs of tetrahedral numbers and is formed by the Young diagrams of stable two-dimensional forms (seen in for example figure 3). The stable plane partition of P_k consists of the stable partition of T_k as the bottom layer, the stable partition of T_{k-1} as the second layer etc.

Lemma 2. *If there exists a cube in one diagonal plane and a hole in a lower diagonal plane, then there exists a sequence of moves which leads to the hole being filled.*

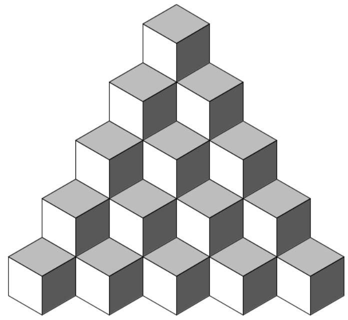
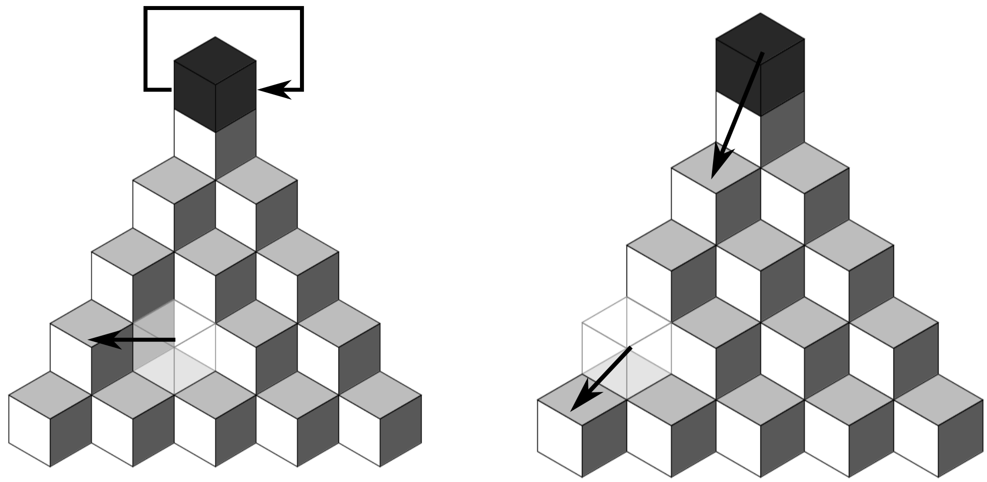


Figure 13: This is the stable form of the tetrahedral number $P_5 = 35$.

Proof. Let there be a cube in one diagonal plane and a hole in a lower one. We shall now construct a sequence of moves fulfilling the criteria given above. This problem can be reduced to moving the cube and the hole to the same layer, because then by lemma 1 there exists a sequence fulfilling the criteria. In order to put them in the same layer, use the six operations to place the cube in the pile of $\pi_{1,1}$. Notice that this place, in every layer parallel to b, corresponds to the first diagonal - the one of length 1. When performing the moves $B(\pi, b)$ and $B'(\pi, b)$ all cubes and holes are cycling on their diagonals parallel to b. That is: the cube in the first diagonal will not change position at all, whilst the hole will move. Just apply $B(\pi, b)$ until they are in the same layer (figure 14a). Then use operations with respect to the side parallel to that layer, until the hole has been filled as by lemma 1 (figure 14b). If during this procedure the hole is prematurely filled by another cube or the cube fills another hole, the lemma is still fulfilled. □

Theorem 6. *For each plane partition π of P_k, there is a path in the game graph from π to the stable state.*

Proof. By lemma 2, it is possible to fill all holes, as long as there are still cubes in higher diagonal planes. Since P_k cubes exactly fill k diagonal planes, after filling all holes the stable state has been achieved. □



(a) Make the cube and the hole go into the same layer.

(b) Perform moves with respect to the side parallel to that layer.

Figure 14: The key ideas of constructing a sequence of moves which fills a hole.

We can now define a game with an objective, based on the three-dimensional Bulgarian Solitaire: *start at an arbitrary plane partition* π *of* P_k*. Use the 6 operations to transform* π *into the stable state of* P_k *in as few moves as possible.* By theorem 6 this game always has a solution, no matter which plane partition of P_k the player starts at.

The game could also be made slightly harder, for example by restricting the allowed operations to $B(\pi, b)$ and $B(\pi, l)$. It can be proved solvable with these limitations, but we leave this proof as an excercise for the reader.

4 Future research

The research on Bulgarian Solitaire in three dimensions has only just begun; there are still many unexplored areas. One interesting question is if there exists any Garden of Eden partitions in the three-dimensional game. I think that it does not, but this is yet to be proven. This conjecture has been confirmed for plane partitions π up to $\sum \pi_{ij} = 9$ through computer simulations made as part of this study.

Conjecture 1. *Using the six defined operations, there are no plane partitions which can not be reached from another plane partition.*

There are also many questions related to the game proposed in the end of section 3.4: is there an optimal strategy to minimize the number of moves needed to change an arbitrary plane partition of a tetrahedral number into the stable state? What is the maximum number of moves needed if playing optimally? Can this be generalized into a competitive multiplayer game?

Also, it would be interesting to investigate how the game would behave if allowing parts of infinite size, or infinitely many parts. Furthermore, could it be possible to define Bulgarian Solitaire on continous functions?

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References

[1] Hopkins B. 30 Years of Bulgarian Solitaire. The College Mathematics Journal. 2012;43(2):135–140.

[2] Eriksson H. Bulgarisk patiens. Elementa. 1981;64(4):186–188.

[3] Brandt J. Cycles of Partitions. Proceedings of the American Mathematical Society. 1982 Jul;85(3):483. Available from: http://www.jstor.org/stable/2043873?origin=crossref.

[4] Stanley RP. Enumerative Combinatorics Volume 2. Cambridge University Press; 1999.