

On the Energisation Mechanism of Charged Particles in ABC Fields

Toomas Liiv
toomas.liiv@outlook.com

under the direction of

Dhrubaditya Mitra
Nordic Institute for Theoretical Physics

Bidya Binay Karak
Nordic Institute for Theoretical Physics

Research Academy for Young Scientists
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Abstract

Cosmic rays have puzzled the world for more than a century and their origin is still not known. Recently, it has been shown that energisation of charged particles in Arnold-Beltrami-Chioldress magnetic fields with coefficients $A = B = C = 1$ and sinusoidal time-dependence is possible. This paper simulates the fields with RK4 and investigates the mechanism behind the energisation process by producing histograms over the energy injections and their persistence in sign, i.e. the time it takes for it to change. The histogram of energy injections is Gaussian-like and skewed to the positive side and the histogram of persistence in sign is possibly log-normal or Gaussian with a power-law tail. Further studies should focus on determining the physical significance of these results and to run the simulation for longer times, and this paper provides a framework for how it could be done.

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1 Introduction

It has recently been shown that energisation of charged particles in chaotic and time-dependent Arnold-Beltrami-Childress (ABC) magnetic fields is possible [1]. This paper describes the mechanism of energisation in further detail, examining the distribution of energy injections and the persistence of their sign, i.e. how long it is positive or negative.

1.1 Background

The topic of this paper originates from the classic *cosmic rays problem*. Cosmic rays, not actually being radiation, consist of charged particles with energies up to 10^{20} eV impacting the surface of the Earth at great velocities [2, 3]. In comparison, the highest energies currently produced in particle accelerators are in the order of 10^{13} eV [4]. The particles include protons, electrons and helium nuclei, as well as other heavier nuclei and more exotic particles, such as antiprotons, positrons and muons [5].

Though discovered as early as 1912 by Hess [3], the question regarding the origin of cosmic rays is still open. Undoubtedly, it is of great importance to understand cosmic rays, and in particular their source. Just as astronomy may use light to study the Universe, astrophysics may use cosmic rays to learn about space. Cosmic rays are also one of the few ways to sample actual matter from places beyond the Solar System, and can therefore be used to, for example, study the nature of supernovas [6].

It was first conjectured that the main source of cosmic rays is the Sun, but as no day-night-time difference could be observed, this was disproved [2]. Moreover, the isotropic nature of the radiation also indicates interstellar, galactic or intergalactic origin [2]. Baade and Zwicky proposed that the origin was supernovas or their remnants [7], though it was first with Fermi's seminal paper that an actual mechanism was put forward [8].

Fermi suggested that the energisation of charged particles to such great energies could happen in a mechanism whereby the charged particles are reflected back and forth between two so called magnetic mirrors, a type of magnetic field that reflects charged particles,

moving towards each other [1, 8].

Through numerical simulations in a simplified setting — a ball bouncing between a rigid and a moving wall with a time-dependent motion — it has been shown that energisation could not happen if the motion of the moving wall was a smooth function of time, i.e. particles could not gain energy if the motion of the wall was differentiable [9]. It was later shown that energisation could happen if the motion of the moving wall was *not* a smooth function of time [10].

That a particle actually can increase in energy, provides a basis of indication that energisation of charged particles in a time-dependent magnetic field is possible, which also was formally conjectured in 2012 by Dasgupta et al. for *chaotic* time-dependent magnetic fields [11]. This was later proven through numerical simulations for the special case of an ABC magnetic field with coefficients $A = B = C = 1$ and sinusoidal time-dependence [1]. It has been shown that the streamlines in this magnetic field are chaotic [12].

ABC magnetic fields are merely theoretical constructions with idealized properties for simple calculations. But they are a good choice of a model because of their helicity, which many of the real-world magnetic fields exhibit as a result of the dynamo mechanism producing them [13], and of course their inherent simplicity in calculations. However, the growth in energy is important, as it could be the basis of a solution to the injection problem of first order Fermi acceleration [1].

1.2 Aim

Mitra et al. showed that after a sufficiently long time, the energy of the particle grows as t^ϵ with ϵ depending on the circular frequency of the time-dependence of the magnetic field [1]. However, the details of the energisation mechanism are not known. This paper investigates it in further detail.

In short, the change with time of the magnetic field creates an electric field. Both fields produce a force on the charged particle, but it is only the component of the electric field in the same direction as the velocity that causes a change in energy of the particle.

We simulate this by numerically solving a system of ordinary differential equations with Runge-Kutta's fourth order method. Specifically, we examine the energy change distribution of the particles by producing a histogram of the energy injections at each time-step in the algorithm. Moreover, the distribution of the persistence of sign is examined. This is similarly done by producing a histogram of the aforementioned data.

2 Model

In this section the equations governing the system and how they are solved is presented. The principal part of the model is the same as described by Mitra et al. [1].

2.1 Theory

The force \mathbf{F} on a particle with charge q in a magnetic field is

$$\mathbf{F} = q \cdot (\mathbf{E} + \mathbf{V} \times \mathbf{B}), \quad (1)$$

where \mathbf{E} denotes the external electric field, \mathbf{B} the magnetic field, \mathbf{V} the velocity of the particle and \times the cross product of two vectors¹. The term $q \cdot (\mathbf{V} \times \mathbf{B})$ is called the magnetic force and the term $q \cdot \mathbf{E}$ the electric force. Together they constitute the *Lorentz force*. It is of importance to notice that the magnetic force is perpendicular to the particles velocity \mathbf{V} , due to the nature of the cross product.

In this problem, \mathbf{B} is time-dependent, i.e., it is a function of time. Specifically

$$\mathbf{B} = \mathcal{B} \cdot \sin(\omega t), \quad (2)$$

where t is time and ω is the circular frequency. The vector $\mathcal{B} = (\mathcal{B}_x, \mathcal{B}_y, \mathcal{B}_z)$ is an ABC

¹The cross product is an operation between two (three-dimensional) vectors defined by $\mathbf{a} \times \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin \theta \cdot \hat{\mathbf{n}}$, where θ is the angle between the two vectors \mathbf{a} and \mathbf{b} and $\hat{\mathbf{n}}$ is a unit vector orthogonal to the plane containing \mathbf{a} and \mathbf{b} . It can also be calculated using determinants, as shown in appendix A.

magnetic field, defined by

$$\begin{cases} \mathcal{B}_x = B_0(A \cdot \sin kz + C \cdot \cos ky) \\ \mathcal{B}_y = B_0(B \cdot \sin kx + A \cdot \cos kz) \\ \mathcal{B}_z = B_0(C \cdot \sin ky + B \cdot \cos kx), \end{cases} \quad (3)$$

where k is the wavenumber of the field, B_0 a constant describing the magnitude and x, y, z positions in three dimensional space.

The electric field \mathbf{E} is generated by the change of the magnetic field \mathbf{B} with time, according to the *Maxwell-Faraday* equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (4)$$

where ∇ is the curl operator and $\frac{\partial}{\partial t}$ the derivative with respect to time. This differential equation has the solution

$$\mathbf{E} = \frac{\omega}{k} \cdot \mathbf{B} \quad (5)$$

because of the property that \mathbf{B} is an eigenfunction of the curl operator².

Mitra et al. has recently shown that in a field described by equations 2 and 3, with coefficients $A = B = C = 1$, particles with $\frac{q}{m} = 1$ can be energized, where q is the charge and m is the mass. The energisation consists of two distinct phases, the second one starting at a time where roughly $\omega t = 100$ [1]. Note that the units are arbitrary, see section 2.2 for further details. This study has only examined times greater than this, since the first phase is believed to be the result of idealized and un-physical initial conditions of the system. That is, at first all particles are uniformly distributed both in position and velocity, which is unlikely in a real-world context. By letting the simulation run past a certain time, this problem is somewhat countered.

How does this energisation process go about? Consider a particle with energy of W .

²The details of the curl operator and this property are provided in appendix A.

The only forces acting on the particle are the electric force \mathbf{F}_E and the magnetic force \mathbf{F}_B . In a linear approximation, as is implemented and described in section 2.2, the force is considered to be constant in some interval where the particle undergoes a displacement s . The difference in energy can then be calculated in accordance with

$$\Delta W = \mathbf{F} \cdot \cos \theta \cdot \Delta s = (|\mathbf{F}_E| \cdot \cos \theta_E + |\mathbf{F}_B| \cdot \cos \theta_B) \cdot \Delta s \quad (6)$$

where θ_E is the angle between the electric field and the velocity of the particle and θ_B is the angle between the magnetic field and the velocity of the particle.

However, as noted in the beginning of this section, the magnetic force is perpendicular to the velocity of the particle. Thus, $\cos \theta_B = \cos \frac{\pi}{2} = 0$. Combined with equation 1, it is thus implied that the difference in energy can be simplified to

$$\Delta W = |\mathbf{F}_E| \cdot \cos \theta_E \cdot \Delta s = q \cdot |\mathbf{E}| \cdot \cos \theta_E \cdot \Delta s. \quad (7)$$

Note that q and s may vary depending on the configuration of the simulation, and an universal value is more interesting. Therefore

$$\frac{\Delta W}{q \cdot s} = |\mathbf{E}| \cdot \cos \theta_E \quad (8)$$

is calculated instead. This is the component of the electric field in the direction of the velocity of the particle and is henceforth referred to as the *energy injection*.

2.2 Computation

Assuming non-relativistic speeds, Newton's second law can be used to rewrite equation 1 into

$$\frac{d^2 \mathbf{x}}{dt^2} = \frac{q}{m} \cdot (\mathbf{E} + \mathbf{V} \times \mathbf{B}), \quad (9)$$

where $\mathbf{x} = (x, y, z)$ is position. This second order differential equation can be split into the two first order differential equations

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{v} \\ \frac{d\mathbf{v}}{dt} = \frac{q}{m} \cdot (\mathbf{E} + \mathbf{V} \times \mathbf{B}), \end{cases} \quad (10)$$

where $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity.

Initially, $N = 1000$ particles with $\frac{q}{m} = 1$ are distributed uniformly³ in a cube with dimensions $2\pi \times 2\pi \times 2\pi$, with velocities uniformly distributed between -0.01 and 0.01 along the x -axis, inside an ABC field as described in equation 2 and 3 with $k = 1$, $B_0 = 1$ and $\omega = 0.1$. These are similar to the values chosen by Mitra et al. [1].

This system of differential equations is solved through a stepping method, specifically numerically integrated with the fourth order Runge-Kutta method, with the fixed step-size of $\Delta t = 0.1$ up to the total time of $t_{\max} = 9118.4$. This provides the position and the velocity throughout the process.

The actual implementation is called PYODEN⁴, based on PYTHON with NUMPY and MATPLOTLIB. For a reference of the algorithm, see *Numerical Recipies in Fortran* [14] or a brief overview in appendix B.

This paper is interested in the component of the electric field in the direction of the velocity, the energy injection. This quantity is calculated through the observation

$$\frac{\Delta W}{q \cdot s} = |\mathbf{E}| \cdot \cos \theta_E = \frac{|\mathbf{E}| \cdot |\mathbf{V}| \cdot \cos \theta_E}{|\mathbf{V}|} = \frac{\mathbf{V} \cdot \mathbf{E}}{|\mathbf{V}|} \quad (11)$$

at all time-steps.

Firstly, the energy injection was plotted against time. Secondly, a histogram with 250 bins was produced, representing a probability density function and a curve fitted to it. Thirdly, a histogram is produced for the persistence of sign⁵, i.e. the time it takes for the

³In reality, they are randomly placed inside the cube to mimic uniformity.

⁴<http://code.google.com/p/pyoden/>, GNU GPL v3 licence

⁵See appendix C for details.

sign of the energy injection to change.

All graphs are non-dimensionalized. The unit length is assumed to be $L = 2\pi k$ and the unit time $T = 2\pi\omega$. The electric field has the unit $\frac{\text{kg}\cdot\text{m}}{\text{s}^2\cdot\text{C}}$. Since we are only considering particles with $\frac{q}{m} = 1$, kg and C can be removed. What is left is $\frac{\text{m}}{\text{s}^2}$, which will have the unit strength $\frac{2\pi k}{(2\pi\omega)^2} = \frac{k}{2\pi\omega^2}$. All measurements are divided by their unit magnitude to remove units. This process is known as *non-dimensionalization*.

All histograms are also normalized, that is, the frequency is modified to make the integral of the histogram equal to 1.

3 Results

The sum of the positive energy injections was 135000 and the sum of the negative energy injections was -132000 . Thus, the total energy injection was 3000, the ratio of positive to negative 1.021:1 and the mean $3.37 \cdot 10^{-5}$. These values are calculated taking all the particles into account.

Figure 1 shows the variation of the energy injection of one particle over time.

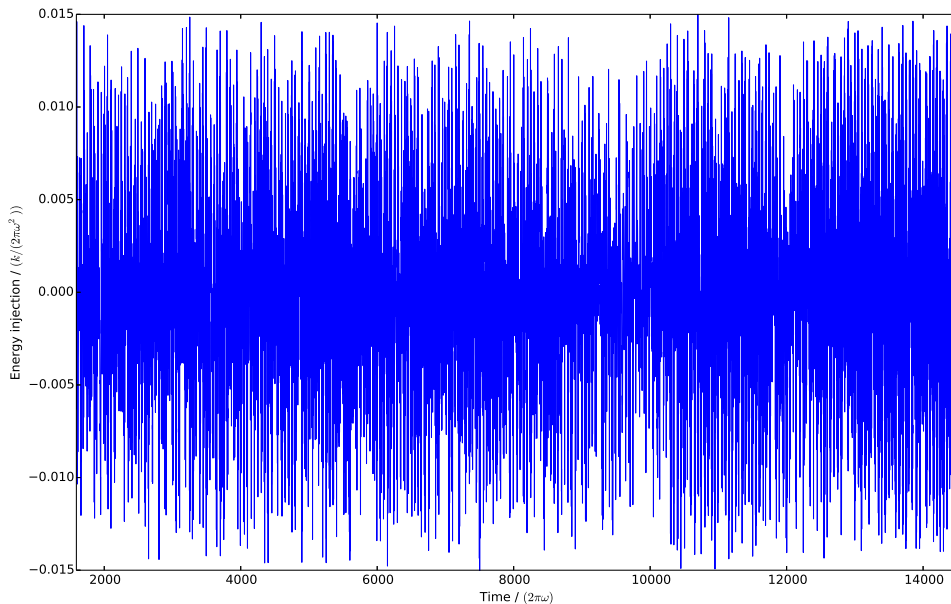


Figure 1: Energy injections plotted against time, non-dimensionalized.

Figure 2 shows a histogram of the energy injections, that is, the frequency of small intervals of energy injections.

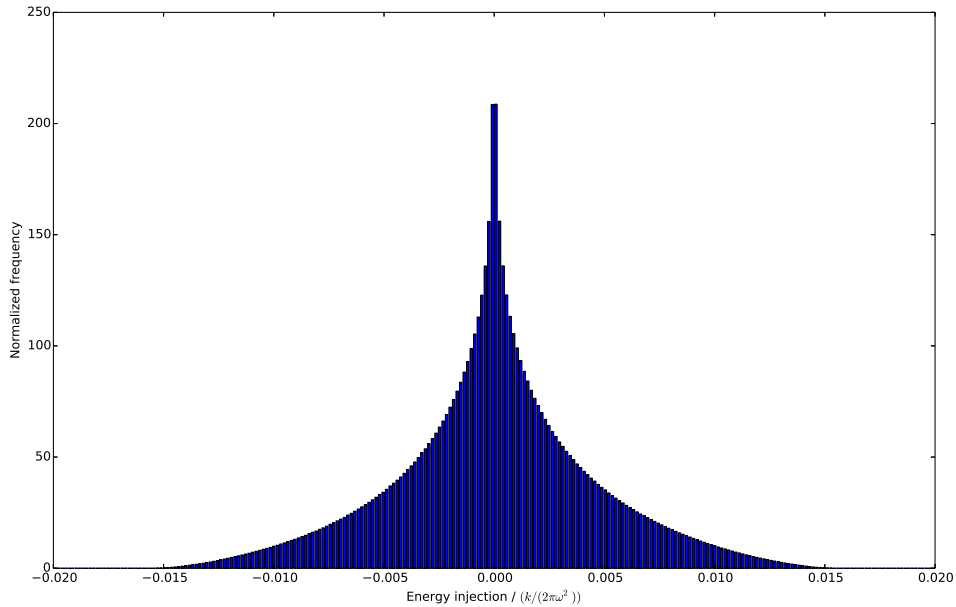


Figure 2: Histogram over the energy injections in the range -0.02 to 0.02 with 250 bins, non-dimensionalized and normalized.

Figure 3 shows two histograms: one of the persistence of the positive sign and the second of the negative sign, i.e., the time for the sign of the energy injection to change.

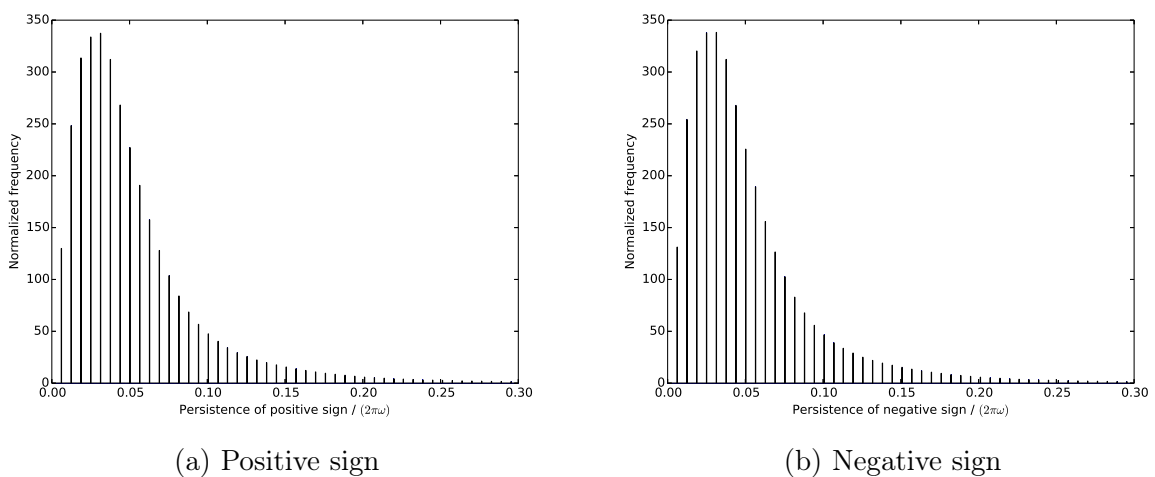


Figure 3: Histogram over the persistence of sign of the energy injections in the interval 0 to 0.03 with 250 bins, non-dimensionalized and normalized.

4 Discussion

From the fact that the ratio between positive and negative energy injections is 1.021:1, we can conclude that the particle does in fact increase its velocity and thus energy as well. This is in line with the results of Mitra et al. [1].

In figure 1, we see the energy injections plotted against time. They are somewhat uniformly distributed around 0, actually somewhat over it, since the energy is increasing. The function does not seem to be smooth in time. This could be because of two reasons: the discrete nature of the algorithm or the chaotic streamlines of the particles. For an actual conclusion, further studies about the differentiability should be done.

Figure 2 shows the distribution of the energy injections. Notice that the peak is somewhat off-set from 0. What is surprising is the smooth and symmetric nature of the it. Why does it to such a great extent resemble a Gaussian? It is known that it is somewhat skewed to the positive side, but the difference is very small, a mere 2.1 %.

Figure 3 shows the the distribution of the persistence of sign. Again, the symmetric nature between positive and negative is striking. Both graphs resemble Gaussians with power-law tails or log-normal distributions.

Persistence of sign is a quantity that describes the "memory" of the system. To what extent does the current state depend on the previous? With this interpretation, the distributions can be explained somewhat physically. The "memory" of the system is small, but it exists and is greater than 0. What is interesting is the power law tail. At what instances does the system remember more than a factor of seven more than the typical case? Is it for small or big energy injections? This is a topic that surely should be investigated further.

Other further investigations on the topic should include the same investigation, but for longer times, preferably to at least $t_{\max} = 100000$, and also look at how the distribution changes with time. It should also be done with a greater accuracy, a $\Delta t = 0.01$, should give better results. Furthermore, investigations of the possible relativistic effects for big

energies should be conducted.

Concluding, what this paper shows, is that the growth in energy is very "smooth". There is no particular value of the energy injection that results in growth of velocity, it is the whole distribution that is uniformly and very slightly skewed to the positive side. It is the average energy injection that is important, not a few specific. Furthermore, the system has roughly the same memory, both for positive and negative energy injections.

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⁶No particular order.

References

- [1] Mitra, D., Brandenburg, A., Dasgupta, B., Niklasson, E., Ram, A., *Particle energization through time-periodic helical magnetic fields*, Physical Review E, **89**, April 2014, P. 042919.
- [2] Erlykin, A. D., Wolfendale, A. W., *Cosmic rays: the centenary of their discovery*. Europhysics News, **43**(2), April 2012, P. 26–28.
- [3] Kachelrieß, M., *Lecture Notes on High Energy Cosmic Rays*, August 2007. Web source. Available at <http://web.phys.ntnu.no/~mika/notes.pdf>. Accessed 27 June, 2014.
- [4] CERN Press Office, *The first LHC protons run ends with new milestone*, December 2012. Web source. Available at <http://press.web.cern.ch/press-releases/2012/12/first-lhc-protons-run-ends-new-milestone>. Accessed 28 June, 2014.
- [5] Beringer, J. et al. (Particle Data Group), *The Review of Particle Physics*. Physical Review D, **86**, July 2012, P. 010001.
- [6] NASA (National Aeronautics and Space Administration), *Why Do We Study Cosmic Rays?*. Web source. Available at http://helios.gsfc.nasa.gov/qa_cr.html#crstudy. Accessed 6 July, 2014.
- [7] Baade, W., Zwicky, F., *Cosmic Rays From Super-Novae*, Proceedings of National Academy of Sciences of the United States of America, **20**(5), May 1934, P. 254–259.
- [8] Fermi, E., *On the Origin of Cosmic Radiation*, Physical Review, **75**(8), January 1949, P. 1169–1174.
- [9] Ulam, S. M., *On Some Statistical Properties of Dynamical Systems*, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy, Meteorology, and Physics, University of California Press, Berkeley, CA., 1961, P. 315–320.
- [10] Lieberman, A. E., Lichtenburg, A. J., *Stochastic and Adiabatic Behavior of Particles Accelerated by Periodic Forces*, Physical Review A, **5**, May 1971, P. 1852–1866.
- [11] Dasgupta, B., Li, G., Li, X., Ram, A., *Particle Transport and Acceleration in a Chaotic Magnetic Field: Implications for Seed Population to Solar Flare and CME*, AIP Conference Proceedings, **1500**, 2012, P. 56–63.
- [12] Dombre, T., Frisch, U., Greene, J. M., Hénon, M., Mehr, A., Soward, A.M., *Chaotic streamlines in the ABC flows*, Journal of Fluid Mechanics, **167**, November 1985, P. 353–391.
- [13] Brandenburg, A., Subramanian, K., *Astrophysical magnetic fields and nonlinear dynamo theory*, Physics Reports, **417**, September 2005.
- [14] Press, W., Flannery, B., Teukolsky, S., Vetterling, W., *Numerical Recipes in Fortran*, Cambridge University Press, Cambridge, 1992.

A The Determination of the Electric Field

We claim that

$$\mathbf{E} = \frac{\omega}{k} \cdot \mathbf{B} = \frac{\omega}{k} \mathbf{B} \cos \omega t \quad (12)$$

is a solution to

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (13)$$

where $\frac{\partial}{\partial t}$ is the derivative with respect to time and ∇ the curl operator as defined by

$$\nabla = \left(\hat{\mathbf{i}} \cdot \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right), \quad (14)$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors in the x , y and z direction respectively.

The solution is easy to verify given the property that the curl operator is the eigenfunction of the ABC-field, or in symbolic form

$$\nabla \times \mathbf{B} = k \cdot \mathbf{B}. \quad (15)$$

Then, we simply get

$$\begin{aligned} \text{LHS} &= \nabla \times \mathbf{E} \\ &= \nabla \times \left(\frac{\omega}{k} \cdot \mathbf{B} \right) \\ &= \nabla \times \left(\frac{\omega}{k} \cdot \mathbf{B} \cos \omega t \right) \\ &= \frac{\omega}{k} \cdot \cos \omega t \cdot (\nabla \times \mathbf{B}) \\ &= \frac{\omega}{k} \cdot \cos \omega t \cdot k \cdot \mathbf{B} \\ &= \omega \cos \omega t \cdot \mathbf{B} \\ &= -\frac{\partial \mathbf{B}}{\partial t} \\ &= \text{RHS}. \end{aligned}$$

By this we have shown that the solution is correct given that the assumption that the ABC function is an eigenfunction of the curl operator holds. Now, let us prove this. We do this by simply going from LHS to RHS and calculating the cross product with the help of determinants.

$$\begin{aligned}
\text{LHS} &= \nabla \times \mathcal{B} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathcal{B}_x & \mathcal{B}_y & \mathcal{B}_z \end{vmatrix} = \hat{\mathbf{i}} \left(\frac{\partial \mathcal{B}_z}{\partial y} - \frac{\partial \mathcal{B}_y}{\partial z} \right) - \hat{\mathbf{j}} \left(\frac{\partial \mathcal{B}_z}{\partial x} - \frac{\partial \mathcal{B}_x}{\partial z} \right) + \hat{\mathbf{k}} \left(\frac{\partial \mathcal{B}_y}{\partial x} - \frac{\partial \mathcal{B}_x}{\partial y} \right) \\
&= \hat{\mathbf{i}} \left(\frac{\partial}{\partial y} (B_0 \cdot (C \cdot \sin ky + B \cdot \cos kx)) - \frac{\partial}{\partial z} (B_0 \cdot (B \cdot \sin kx + B \cdot \cos kz)) \right) \\
&\quad - \hat{\mathbf{j}} \left(\frac{\partial}{\partial x} (B_0 \cdot (C \cdot \sin ky + B \cdot \cos kx)) - \frac{\partial}{\partial z} (B_0 \cdot (A \cdot \sin kz + C \cdot \cos ky)) \right) \\
&\quad + \hat{\mathbf{k}} \left(\frac{\partial}{\partial x} (B_0 \cdot (B \cdot \sin kx + A \cdot \cos kz)) - \frac{\partial}{\partial y} (B_0 \cdot (A \cdot \sin kz + C \cdot \cos ky)) \right) \\
&= \hat{\mathbf{i}} \cdot (B_0 \cdot C \cdot k \cdot \cos ky + B_0 \cdot A \cdot k \cdot \sin kz) \\
&\quad - \hat{\mathbf{j}} \cdot (-B_0 \cdot B \cdot k \cdot \sin kx - B_0 \cdot A \cdot k \cdot \cos kz) \\
&\quad + \hat{\mathbf{k}} \cdot (B_0 \cdot B \cdot k \cdot \cos kx + B_0 \cdot C \cdot k \sin ky) \\
&= k \cdot (\hat{\mathbf{i}} \cdot (B_0(A \cdot \sin kz + C \cdot \cos ky)) \\
&\quad + \hat{\mathbf{j}} \cdot (B_0(B \cdot \sin kx + A \cdot \cos kz)) \\
&\quad + \hat{\mathbf{k}} \cdot (B_0(C \cdot \sin ky + B \cdot \cos kx))) \\
&= k \cdot (\mathcal{B}_x, \mathcal{B}_y, \mathcal{B}_z) = k \cdot \mathcal{B} = \text{RHS}
\end{aligned}$$

□

B Runge-Kutta Fourth Order Method

We wish to solve an equation of the type

$$\frac{dy}{dt} = f(y, t). \quad (16)$$

This means that we wish to find an y such that $y = g(t)$, that is, a function of time.

A common approximation is the *Euler method*, which simply states that for small step-size h ,

$$y(t_1 + h) \approx y(t_1) + h \cdot f(y, t_1) \quad (17)$$

at a time t_1 .

The Runge-Kutta 4th order method is very similar to this. The difference is in the slope. Here, just $f(y, t_1)$ is used, but in RK4 it is an weighted average of 4 (hence the order) different slopes.

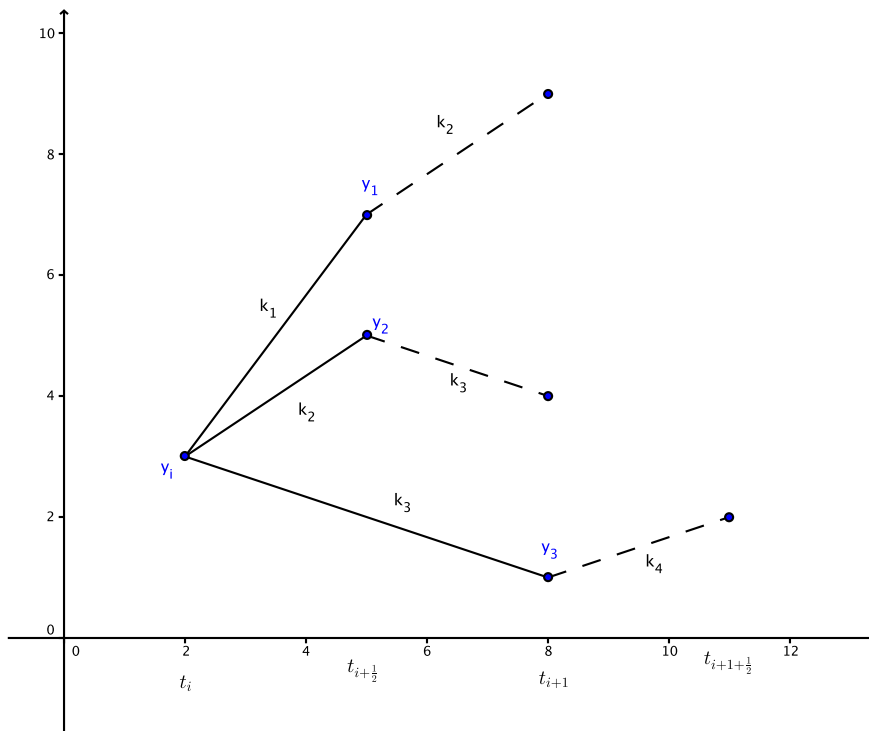


Figure 4: Illustration of RK4.

The RK4 method can roughly and intuitively be described by figure 4 and the following process, where h denotes the step-length of the algorithm:

1. From y_i , evaluate the slope, k_1 and take half a step forward to y_1 . Then $y_1 = y_i + k_1 \cdot \frac{1}{2} \cdot h$.
2. Evaluate the slope at y_1 to k_2 .
3. From y_1 , take half a step forward with slope k_2 to y_2 . Then $y_2 = y_i + k_2 \cdot \frac{1}{2} \cdot h$.
4. At y_2 , evaluate the slope to k_3 .
5. From y_1 , take a whole step forward with slope k_3 to y_3 . Then $y_3 = y_i + k_3 \cdot h$.
6. Evaluate the slope at y_3 to k_4 .
7. Then $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$.

Why the weighed average is the right one is not trivial, but the fact that a weighed average can be better than the Euler method should be no surprise, simply because more slopes in the same interval are taken into account.

C Persistence of Sign

For a function dependent of time, $f(t)$, persistence of sign is the difference of time for an interval where the sign is persistent.

For example, for the function below, the positive persistence of sign is $\Delta t = 6$ and the negative persistence of sign is $\Delta t = 2$.

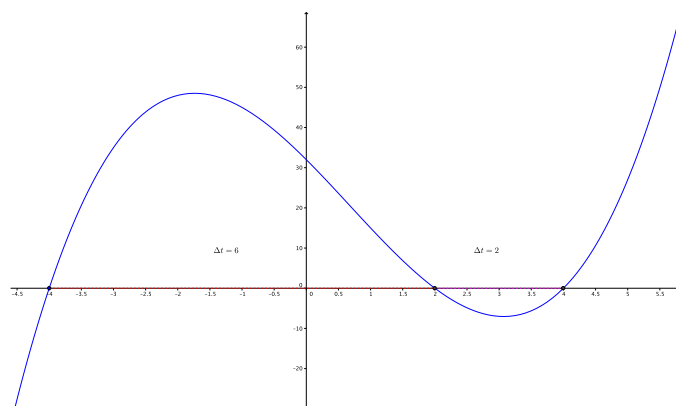


Figure 5: Illustration of persistence in sign.