Approximating a General Function to Compute Wilson Loops Within a Toy Model

Lawan Fathullah
lafa.pcm@procivitas.se

under the direction of
Dr. Edorado Vescovi
Theoretical physics
Nordic Institute for Theoretical Physics

Research Academy for Young Scientists
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Abstract

Electrically charged particles can still be affected by an electromagnetic field, even if there is no electric or magnetic field prevalent at that position. This is called the Aharonov-Bohm effect and was described by Yakir Aharonov and David Bohm in a paper written in 1959. The paper states that an electromagnetic potential can still affect the particles even in areas with no electric or magnetic field. It outlined an experiment that would test this phenomenon, which in later years was conducted and found to support Aharonov’s and Bohm’s paper. The effect that electromagnetic potential has on electrons can be calculated using Wilson loops. Moreover, the strength of interactions between quarks, strength of interactions between electrons and electric potential can all be quantified using Wilson loops. This paper forms and simplifies an integro-differential equation for a function that can be used to calculate Wilson loops. The final result is a program that can provide an approximation of the function, which also serves as groundwork for further calculations.
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1 Introduction

In quantum mechanics, the position of an electron, among other particles, at a certain time is described by a wave function, rather than being a fixed point in space-time [1]. Just like an object in Newtonian physics, a wave function is affected by different forces acting on it. The principle of locality states that a particle, and therefore the wave function, can only be affected by the surroundings of that particle, meaning that there must be something directly in contact with the particle for it to be influenced in any way. For example, consider two electrically charged particles that are not in contact with each other. Each of these particles create an electric field. The charged particles are in direct contact with the electric field of the other particle, allowing the fields to act with a force on the particles [1].

1.1 Aharonov-Bohm Effect

In 1959, Yakir Aharonov and David Bohm published a paper that predicted an effect of electric potentials on charged particles, despite there being no considerable electric or magnetic field to affect the charged particles, which would oppose the principle of locality [2]. The paper outlined an experiment to test this, including a coil of metal called a solenoid. The experiment is very similar to a double-slit experiment, see figure 1. A ray of identical electrons split into two different rays at point a in figure 1. Each ray passes through each slit in a double-slit plate and a screen is used to detect and compile an interference pattern. Behind the double-slit plate is the solenoid that the ray of electrons pass around. This solenoid does not affect the electrons in any way as long as there is no electric current running through it, leaving all the electrons identical. Running an electric current through the solenoid creates a magnetic field inside of the solenoid, but the magnetic and electric field outside of the solenoid is so small that it is negligible. This implies that the electrons would remain unaffected and that the interference pattern would largely stay the same, but observing the interference pattern while a current
is running through the solenoid shows that the electrons have been affected significantly [3].

Figure 1: Illustration of the Aharonov-Bohm effect. The yellow trail shows the path that electrons travel when there is no electric current running through the solenoid. The red trail shows the path that electrons travel when there is an electric current in the solenoid [4].

This reinforced Aharonov’s and Bohm’s paper and showcased the error with describing electromagnetic fields solely as an electric and magnetic field. It would in this case oppose the principle of locality since there would be no field to act on the particles [2]. To preserve the principle of locality, the electromagnetic field must be described by the electromagnetic four-potential. Calculating the electromagnetic field using the electromagnetic four-potential proves that there still is an electromagnetic potential that the electrons can be affected by [5]. That is more in accord with the predictions of Aharonov’s and Bohm’s paper and the principle of locality.

1.2 Quantum Chromodynamics

Quarks are elementary particles that combine to create protons and neutrons. In Quantum chromodynamics, QCD, quarks can have one of three colour charges, blue, red and green. Furthermore, there are antiquarks which can each have the three opposite charges
antiblue, antired and antigreen [6]. For a proton or neutron to be formed, three quarks, each with a different colour charge must combine, creating a "white" particle [6]. Quarks interact and stay bound together by exchanging colour charges, mediated by an exchange particle called the gluon [6]. This exchange is called the strong nuclear force and is illustrated by figure 2. In normal QCD, the number of colour charges, $N$, is equal to 3.

Figure 2: An illustration of the interactions within a proton. The blue up quark generates a gluon which contains a blue and an antigreen charge. This causes the up quark to lose the original blue charge, but the antigreen charge in the gluon creates in turn a green charge within the up quark. The down quark receives the gluon, leading to the antigreen charge neutralising the pre-existing green charge and the blue charge being transferred into it.

In addition, there’s a coupling constant, $\lambda$, that describes the strength of interactions between quarks. There are different physical theories and toy models where the values for $N$ and $\lambda$ vary, such as $N = 4$ and $N = 2^*$ supersymmetric Yang-Mills (SYM) theories [7]. The SYM theories are simpler physical models that are not representative of the real world, but they may provide a starting point for more complicated and accurate theories [7].
1.3 Wilson Loops

A Wilson loop is defined by a quantity on a closed curve in electromagnetism. To concretise, it is able to measure the effect of the electromagnetic potential on electrons even when there is no electric or magnetic field such as in the experiment described above [8]. More uses to Wilson loops include calculations in QCD and SYM theories, more specifically calculating $\lambda$, and computing the electric potential within an electric field [8].

1.4 Aim of Paper

The aim of this paper is to formulate an integro-differential equation for the function $W$, to then simplify it and solve it with the help of a computer program. The solution can then be used to describe a general Wilson loop by calculating $W(2\pi)$. For this particular paper, we will be solving the equation in a toy model where $N \to \infty$ and $\lambda \to 0$.

2 Calculations

A general Wilson loop can be defined as

$$W(2\pi) = \frac{1}{N} \left\langle \sum_i e^{2\pi a_i} \right\rangle$$

according to [9]. We will be studying a more general function $W(x)$. It will be easier to take this more general approach before setting $x = 2\pi$ to compute the Wilson loop. $W(x)$ is defined as

$$W(x) = \frac{1}{N} \left\langle \sum_i e^{2\pi a_i} \right\rangle.$$  

For two parameters, the function is defined as

$$W(x, y) = \left\langle \sum_{i,j} e^{x a_i} e^{y a_j} \right\rangle - N^2 W(x) W(y).$$
The angular brackets, \( \langle \ldots \rangle \), are a placeholder for an average over \( a_i \):

\[
\langle \ldots \rangle = \frac{1}{Z} \int \left[ \prod_i da_i \right] \langle \ldots \rangle \exp \left( -N \sum_i V(a_i) \right) \left[ \prod_{i<j} \mu(a_i - a_j) \right]
\]

where

\[
Z = \int \left[ \prod_i da_i \right] \exp \left( -N \sum_i V(a_i) \right) \left[ \prod_{i<j} \mu(a_i - a_j) \right].
\]

Note that \( 1 \leq i, j \leq N \) if no bounds are specified. The multiple integral \( \int \left[ \prod_i da_i \right] \) represents an integral from \(-\infty\) to \(\infty\) over every \(a_i\). \( V \) is a Taylor series which has mathematical properties similar to that of physical potentials, but it does not represent any real physical potential. It is defined as

\[
V(z) = \frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m z^m,
\]

and \( \exp \) is defined as

\[
\exp(x) = e^x.
\]

The function \( \mu \) is called a measure and is given in a specific model. It is always even, meaning that \( \mu(x) = \mu(-x) \).

### 2.1 Forming an equation for \( \mathcal{W} \)

Although computing \( \mathcal{W}(2\pi) \) is theoretically possible with the definition given in (1), it is a rigorous process. Instead, we will form an equation that solves for \( \mathcal{W} \) in hopes of finding a less inefficient formula for \( \mathcal{W}(2\pi) \).

First, we make the assumption that

\[
e^{x a_p} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \to 0
\]

when \( N \to \infty \) and \( a_p \to \pm \infty \). Note that \( 1 \leq p \leq N \), meaning that it is contained within the sum, \( \sum_i \), and product, \( \prod_{i<j} \), of the expression. This assumption allows us to form
the following equation:

\[ 0 = \frac{1}{Z} \int d a_i \sum_p \frac{\partial}{\partial a_p} \left[ e^{x \alpha_p} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \right] \]  

(6)

This is because if \( f(x) \to 0 \) when \( x \to \pm \infty \), then the following is true:

\[ \int_{-\infty}^{\infty} f'(x) \, dx = \lim_{a \to \infty} \int_{-a}^{a} f'(x) \, dx = \lim_{a \to \infty} \left[ f(x) \right]_{-a}^{a} = \lim_{a \to \infty} (f(a) - f(-a)) = 0. \]

In total, it proves that \( \int_{-\infty}^{\infty} f'(x) \, dx = 0 \) in this case. Applying this to (5) gives (6), but not without certain restrictions on \( \mu \).

The derivative of a product of three functions can be written as

\[ \frac{d}{dx} (fgh) = f'gh + fg'h + fgh'. \]

Similarly to this, the partial derivative, \( \frac{\partial}{\partial a_p} \), in (6) can be separated into three different terms where \( f = e^{x \alpha_p}, g = \exp \left( -N \sum_i V(a_i) \right) \) and \( h = \prod_{i<j} \mu(a_i - a_j) \). Separating the partial derivative, we get

\[ 0 = \frac{1}{Z} \int d a_i \sum_p \frac{\partial}{\partial a_p} \left[ e^{x \alpha_p} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \right] \\
= \frac{1}{Z} \int d a_i \sum_p \frac{\partial}{\partial a_p} \left( e^{x \alpha_p} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \right) \\
+ \frac{1}{Z} \int d a_i \sum_p e^{x \alpha_p} \frac{\partial}{\partial a_p} \left( \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \right) \\
+ \frac{1}{Z} \int d a_i \sum_p e^{x \alpha_p} \exp \left( -N \sum_i V(a_i) \right) \frac{\partial}{\partial a_p} \left( \prod_{i<j} \mu(a_i - a_j) \right). \]  

(7)

The next step is to look at the derivative in each term separately and to compute it. For
the first term, the derivative is very simple.

\[ \frac{\partial}{\partial a_p} (e^{xa_p}) = xe^{xa_p} \]  

(8)

To derive the second term, an expansion of the sum is required.

\[ \frac{\partial}{\partial a_p} \exp \left( -N \sum_i V(a_i) \right) = \frac{\partial}{\partial a_p} \left( e^{-NV(a_1)} \cdot e^{-NV(a_2)} \cdots e^{-NV(a_p)} \cdots e^{-NV(a_N)} \right) \]

\[ = e^{-NV(a_1)} \cdot e^{-NV(a_2)} \cdots (-NV'(a_p))e^{-NV(a_p)} \cdots e^{-NV(a_N)} \]

\[ = -NV'(a_p) \exp \left( -N \sum_i V(a_i) \right) \]  

(9)

For the last term, we utilise logarithmic derivation to avoid deriving a product of functions.

\[ \frac{\partial}{\partial a_p} \left( \prod_{i<j} \mu(a_i - a_j) \right) = \left( \prod_{i<j} \mu(a_i - a_j) \right) \frac{\partial}{\partial a_p} \ln \left( \prod_{i<j} \mu(a_i - a_j) \right) \]

\[ = \left( \prod_{i<j} \mu(a_i - a_j) \right) \frac{\partial}{\partial a_p} \left( \sum_{p<i} \ln(\mu(a_p - a_i)) + \sum_{j<p} \ln(\mu(a_j - a_p)) \right) \]

\[ = \left( \prod_{i<j} \mu(a_i - a_j) \right) \frac{\partial}{\partial a_p} \left( \sum_{p<i} \ln(\mu(a_p - a_i)) + \sum_{j<p} \ln(\mu(a_j - a_p)) \right) \]

Since \( \mu \) is an even function, we can do the simplification

\[ \left( \prod_{i<j} \mu(a_i - a_j) \right) \frac{\partial}{\partial a_p} \left( \sum_{i \neq p} \ln(\mu(a_p - a_i)) \right) \],

where the derivative can be computed by using the chain rule \( \frac{d}{dx} (f(g(x))) = \frac{df}{dg} \cdot \frac{dg}{dx} \), where
\( f(x) = \ln(x) \) and \( g(x) = \mu(x) \).

\[
\left( \prod_{i<j} \mu(a_i - a_j) \right) \left( \sum_{i \neq p} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \right)
\]

In total we get that

\[
\frac{\partial}{\partial a_p} \left( \prod_{i<j} \mu(a_i - a_j) \right) = \left( \prod_{i<j} \mu(a_i - a_j) \right) \left( \sum_{i \neq p} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \right).
\]  

(10)

Inserting (8), (9) and (10) into (7), we receive

\[
0 = \frac{1}{\mathcal{Z}} \int \prod_i da_i \sum_p e^{xa_p} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) +
\]

\[
\frac{1}{\mathcal{Z}} \int \prod_i da_i \sum_p e^{xa_p} \left( -NV'(a_p) \exp \left( -N \sum_i V(a_i) \right) \right) \prod_{i<j} \mu(a_i - a_j) +
\]

\[
\frac{1}{\mathcal{Z}} \int \prod_i da_i \sum_p e^{xa_p} \exp \left( -N \sum_i V(a_i) \right) \left( \prod_{i<j} \mu(a_i - a_j) \right) \left( \sum_{i \neq p} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \right)
\]

Using the definition of \( \langle \ldots \rangle \), a further simplification can be made.

\[
0 = \langle x \sum_p e^{xa_p} \rangle - N \langle \sum_p V'(a_p) e^{xa_p} \rangle + \langle \sum_{i \neq p} e^{xa_p} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \rangle
\]

The first two terms can be written in terms of \( \mathcal{W}(x) \). Since \( \mathcal{W}(x) = \frac{1}{N} \langle \sum_i e^{xa_i} \rangle \), the first term can be rewritten according to

\[
\langle x \sum_p e^{xa_p} \rangle = Nx \mathcal{W}(x).
\]

(11)

To rewrite the second term in terms of \( \mathcal{W}(x) \), the definition for \( V' \) must be used.

\[
V'(z) = \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} mT_m z^{m-1}
\]

(12)
Inserting (12) into \(-N \left\langle \sum_p V'(a_p) e^{x a_p} \right\rangle\), the expression

\[-N \left\langle \sum_p \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} m T_m a_p^{m-1} e^{x a_p} \right\rangle\]

is received, which can be simplified further:

\[-N \left\langle \sum_p \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} m T_m \left( \frac{d}{dx} \right)^{m-1} e^{x a_p} \right\rangle = \left\langle \sum_p \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} m T_m \left( \frac{d}{dx} \right)^{m-1} e^{x a_p} \right\rangle = -N V'(\frac{d}{dx}) \left( \sum_p e^{x a_p} \right) = -N V'(\frac{d}{dx}) \mathbb{W}(x). \quad (13)\]

Lastly, the third term can also be rewritten like

\[
\left\langle \sum_{i \neq p} e^{x a_p} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \right\rangle = \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{x a_p} - e^{x a_i}}{a_p - a_i} \right\rangle
\]

\[
= \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{x a_p} - e^{x a_i}}{2} \right\rangle
\]

\[
= \frac{1}{2} \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{x a_p} - e^{x a_i}}{a_p - a_i} \right\rangle,
\]

where

\[
\gamma(x) = x \frac{\mu'(x)}{\mu(x)}.
\]

To rewrite (14) in terms of \(\mathbb{W}(x)\), a substitution must be made. We can do this by using the integral

\[
\int_0^x ds e^{sa_p + (x-s)a_i}.
\]

First, it must be rewritten into a form that can be used in (14).

\[
\int_0^x ds e^{sa_p + (x-s)a_i} = e^{xa_i} \int_0^x ds e^{sa_p - a_i} = e^{xa_i} \left[ \frac{e^{s(a_p - a_i)}}{a_p - a_i} \right]_0^x
\]
\[ e^{xa_i} e^{x(a_p-a_i)} - 1 \quad \frac{e^{xa_p} - e^{xa_i}}{a_p - a_i}. \]

Now the total equation is

\[ \int_0^x ds e^{sa_p+(x-s)a_i} = e^{xa_p} - e^{xa_i}. \]...

Substituting \( \frac{e^{xa_p} - e^{xa_i}}{a_p - a_i} \) in (14) with the left-hand side of (15), we receive

\[ \frac{1}{2} \left< \sum_{i \neq p} \gamma(a_p - a_i) \int_0^x ds e^{sa_p+(x-s)a_i} \right> \]...

We set the Fourier transform of \( \gamma(a_p - a_i) \) of a new function, \( \tilde{\gamma} \),

\[ \gamma(a_p - a_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(a_p-a_i)} \tilde{\gamma}(\omega), \]...

where

\[ \tilde{\gamma} = \int_{-\infty}^{\infty} dx e^{i\omega} \gamma. \]...

By inserting (17) into (16), we receive

\[ \frac{1}{2} \left< \sum_{i \neq p} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(a_p-a_i)} \tilde{\gamma}(\omega) \int_0^x ds e^{sa_p+(x-s)a_i} \right> \]

The expression can be simplified further:

\[ \frac{1}{2} \left< \sum_{i \neq p} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(a_p-a_i)} \tilde{\gamma}(\omega) \int_0^x ds e^{sa_p+(x-s)a_i} \right> \]

\[ = \frac{1}{2} \left< \sum_{i \neq p} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(a_p-a_i)} \tilde{\gamma}(\omega) \int_0^x ds e^{sa_p+xa_i-sa_i} \right> \]
\[
\begin{align*}
&\quad = \frac{1}{2} \left< \sum_{i \neq p} e^{x_{ai}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds e^{s(a_{p} - a_{i})} e^{-i\omega(a_{p} - a_{i})} \right> \\
&= \frac{1}{2} \left< \sum_{i \neq p} e^{x_{ai}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds e^{s(a_{p} - a_{i})} e^{-i\omega(a_{p} - a_{i})} \right> - \frac{1}{2} \left< \sum_{i = p} e^{x_{ai}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \right>
\end{align*}
\]

Because \( \gamma(0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \), assuming that \( \gamma(0) = 2 \) will create a term consisting of \(-Nx\mathbb{W}(x)\), which will be useful for simplifying even more. We insert that \( \gamma(0) = 2 \).

\[
\begin{align*}
&\quad = \frac{1}{2} \left< \sum_{i \neq p} e^{x_{ai}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds e^{s(a_{p} - a_{i})} e^{-i\omega(a_{p} - a_{i})} \right> - N_{x}\mathbb{W}(x) \\
&= \frac{1}{2} \left< \sum_{i \neq p} e^{x_{ai}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds e^{a_{i}(x-s+i\omega)} e^{a_{p}(s-i\omega)} \right> - N_{x}\mathbb{W}(x)
\end{align*}
\]

Using (3), we can write the expression in terms of \( \mathbb{W} \).

\[
\begin{align*}
&\quad = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \\
&\quad \cdot \int_{0}^{x} ds \left[ \mathbb{W}(x - s + i\omega, s - i\omega) + N^{2}\mathbb{W}(x - s + i\omega)\mathbb{W}(s - i\omega) \right] - N_{x}\mathbb{W}(x)
\end{align*}
\]

(18)

Using (11), (13) and (18), the equation can be written as

\[
0 = N_{x}\mathbb{W}(x) - N^{2}V' \left( \frac{d}{dx} \right) \mathbb{W}(x)
\]

\[
\begin{align*}
&\quad \cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds \left[ \mathbb{W}(x - s + i\omega, s - i\omega) + N^{2}\mathbb{W}(x - s + i\omega)\mathbb{W}(s - i\omega) \right] - N_{x}\mathbb{W}(x)
\end{align*}
\]
The $Nx\mathcal{W}(x)$ and $-Nx\mathcal{W}(x)$ terms cancel each other out.

$$0 = -N^2V'\left(\frac{d}{dx}\right)\mathcal{W}(x)$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds \left[ \mathcal{W}(x-s+i\omega, s-i\omega) + N^2\mathcal{W}(x-s+i\omega)\mathcal{W}(s-i\omega) \right]$$

$$= N^2V'\left(\frac{d}{dx}\right)\mathcal{W}(x)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds \left[ \mathcal{W}(x-s+i\omega, s-i\omega) + N^2\mathcal{W}(x-s+i\omega)\mathcal{W}(s-i\omega) \right]$$

Using the definition for $V'$, (12), we receive

$$N^2 \frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_{p}\left(\frac{d}{dx}\right)^{p-1}\mathcal{W}(x)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds \left[ \frac{1}{N^2}\mathcal{W}(x-s+i\omega, s-i\omega) + \mathcal{W}(x-s+i\omega)\mathcal{W}(s-i\omega) \right]$$

Next, we divide the equation by $N^2$.

$$\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_{p}(\frac{d}{dx})^{p-1}\mathcal{W}(x)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds \left[ \frac{1}{N^2}\mathcal{W}(x-s+i\omega, s-i\omega) + \mathcal{W}(x-s+i\omega)\mathcal{W}(s-i\omega) \right]$$

We are studying a toy model where $N \to \infty$, meaning that $\frac{1}{N^2} = 0$. That means that the equation can be written as

$$\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_{p}(\frac{d}{dx})^{p-1}\mathcal{W}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds \mathcal{W}(x-s+i\omega)\mathcal{W}(s-i\omega).$$

(19)
### 2.2 Defining an ansatz

To continue, the ansatz
\[
\mathcal{W}(x) = e^{\tilde{\alpha}x} \mathcal{W}(x)
\]

and
\[
\mathcal{W}(x,y) = e^{\tilde{\alpha}(x+y)} \mathcal{W}(x,y)
\]

will be made. The function \(\mathcal{W}\) is defined as
\[
\mathcal{W}(x) = \tilde{c}_{000} + \sum_{k=0}^{\infty} \sum_{n=2k}^{\infty} \sum_{m=1}^{2n} \tilde{c}_{k,n,m} N^{-2k} \lambda^n x^m
\]

and
\[
\mathcal{W}(x,y) = \sum_{k=0}^{\infty} \sum_{n=2k+1}^{\infty} \sum_{m=1}^{2n-2k} \sum_{m'=1}^{2n-1} \tilde{d}_{k,n,m,m'} N^{-2k} \lambda^n x^m y^{m'},
\]

both of which are Taylor expansions that are meant to approximate the solution to the integro-differential equation. After inserting the ansatz, the equation is now
\[
\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_p \left( \frac{d}{dx} \right)^{p-1} \left( e^{\tilde{\alpha}x} \mathcal{W}(x) \right) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \tilde{\gamma}(\omega) \int_0^x ds N^2 e^{\tilde{\alpha}(x-s+i\omega)} \mathcal{W}(x-s+i\omega) e^{\tilde{\alpha}s-i\omega} \mathcal{W}(s-i\omega).
\]

The expression \(\left( \frac{d}{dx} \right)^{p-1} \left( e^{\tilde{\alpha}x} \mathcal{W}(x) \right)\) is a derivative of a product and can be rewritten according to the Leibniz rule.

\[
\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_p \left( \frac{d}{dx} \right)^{p-1} \left( e^{\tilde{\alpha}x} \mathcal{W}(x) \right)
\]

\[
\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_p \sum_{i=0}^{p-1} \binom{p-1}{i} \left( \frac{d}{dx} \right)^i e^{\tilde{\alpha}x} \left( \frac{d}{dx} \right)^{p-1-i} \mathcal{W}(x)
\]

13
The $i$:th derivative of $e^{\bar{a}x}$ is equal to $a^i e^{\bar{a}x}$, allowing us to break out $e^{\bar{a}x}$.

$$e^{\bar{a}x} \frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_p \sum_{i=0}^{p-1} \binom{p-1}{i} \bar{a}^i \left(\frac{d}{dx}\right)^{p-1-i} \tilde{W}(x)$$

Using the binomial theorem, $(\binom{p-1}{i}) \bar{a}^i \left(\frac{d}{dx}\right)^{p-1-i}$ can be rewritten as $(\bar{a} + \frac{d}{dx})^{p-1}$, giving us

$$e^{\bar{a}x} \frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} pT_p \left(\bar{a} + \frac{d}{dx}\right)^{p-1} \tilde{W}(x).$$

In terms of $V'$, the expression would be written as

$$e^{\bar{a}x} V' \left(\bar{a} + \frac{d}{dx}\right) \tilde{W}(x). \quad (22)$$

To remove $\bar{a}$ from $V' \left(\bar{a} + \frac{d}{dx}\right)$, we create a new function $\tilde{V}$, defined as

$$\tilde{V}(x) = V(\bar{a} + x) = \frac{8\pi^2}{\lambda} \sum_{p=0}^{\infty} \tilde{T}_p x^p,$$

where $\tilde{T}_p$ is a constant that depends on $T_p$ and $\bar{a}$. The definition of $\tilde{V}$ can be used to rewrite (22) as

$$e^{\bar{a}x} \tilde{V}' \left(\frac{d}{dx}\right) \tilde{W}(x).$$

This expression is equal to the left-hand side of (21). Substituting it into the equation, we receive

$$e^{\bar{a}x} \tilde{V}' \left(\frac{d}{dx}\right) \tilde{W}(x) = e^{\bar{a}x} \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \gamma(\omega) \int_{0}^{x} ds \tilde{W}(s-i\omega) \tilde{W}(x-s+i\omega).$$

Dividing the equation by $e^{\bar{a}x}$, the equation is now

$$\tilde{V}' \left(\frac{d}{dx}\right) \tilde{W}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \gamma(\omega) \int_{0}^{x} ds \tilde{W}(s-i\omega) \tilde{W}(x-s+i\omega).$$
2.3 Inserting the definition of \( \tilde{W} \)

Lastly, we insert the definition of \( \tilde{W} \) for one parameter, (20).

\[
\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} p\tilde{T}_p \left( \frac{d}{dx} \right)^{p-1} \left( \tilde{c}_{000} + \sum_{k=0}^{\infty} \sum_{n=2k}^{\infty} \sum_{m=1}^{2n} \tilde{c}_{k,n,m} N^{-2k} \lambda^n x^m \right)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\gamma}(\omega) \int_{0}^{x} ds \left( \tilde{c}_{000} + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=1}^{2n} \tilde{c}_{k,n,m} N^{-2k} \lambda^n (x - s + i\omega)^m \right)
\]

\[
\cdot \left( \tilde{c}_{000} + \sum_{k=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{m'=1}^{2n'} \tilde{c}_{k,n',m'} N^{-2k} \lambda^n (s - i\omega)^m \right)
\]

Again, the fact that \( N \to \infty \) can be utilised. This makes the terms where \( k = 0 \) the only relevant ones because \( N^{-2k} \to 0 \) when \( k \neq 0 \). Using this, we rewrite the equation as

\[
\frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} p\tilde{T}_p \left( \frac{d}{dx} \right)^{p-1} \left( \tilde{c}_{000} + \sum_{n=0}^{\infty} \sum_{m=1}^{2n} \tilde{c}_{0,n,m} \lambda^n x^m \right)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\gamma}(\omega) \int_{0}^{x} ds \left( \tilde{c}_{000} + \sum_{n=0}^{\infty} \sum_{m=1}^{2n} \tilde{c}_{0,n,m} \lambda^n (x - s + i\omega)^m \right)
\]

\[
\cdot \left( \tilde{c}_{000} + \sum_{n'=0}^{\infty} \sum_{m'=1}^{2n'} \tilde{c}_{0,n',m'} \lambda^n (s - i\omega)^m \right). \tag{23}
\]

2.4 Approximating \( \tilde{W} \) using Mathemática

With the finished equation, (23), the value of the constants, \( \tilde{c} \), can be computed using a program written in Wolfram Mathemática, see Appendix A. The amount of constants computed depends on the order of \( \lambda \), which can be adjusted by changing the value of a variable, cutoff\( \lambda \), within the code. For \( \lambda^1 \), we receive

\[
\tilde{c}_{0,1,1} = \frac{3\tilde{T}_3}{64\pi^3 \tilde{T}_2^2}, \quad \tilde{c}_{0,1,2} = \frac{-1}{64\pi^3 \tilde{T}_2}.
\]
3 Discussion

3.1 Issues

The constants $\tilde{c}$ for higher orders of $\lambda$ could theoretically be calculated with the method used, but running the Mathematica program for higher orders would require more computational power or a more efficient program to receive an output within a reasonable time frame. The code can also not be used for physical models where $N$ is a constant, such as QCD where $N = 3$. This means that the final program is still far away from being used to describe Wilson loops in real physical models.

3.2 Conclusion

An integro-differential equation for the function $\mathcal{W}$ was derived. This equation was further simplified in a toy model where $N \to \infty$ and $\lambda \to 0$. After making an ansatz, an equation for the Taylor expansion $\tilde{\mathcal{W}}$ was written. This further simplified the problem into an algebraic equation, which was able to be solved using a program written in Wolfram Mathematica, giving values for the constants within the Taylor expansion, $\tilde{c}$, and in turn allowing us to estimate $\mathcal{W}$. The nature of Mathematica allows for computation of symbolic mathematics. This means that the code can easily be adapted by, for example, inserting a definition for $\mu$. Because of this, the program can serve as a base for more complicated physical theories.

3.3 Further Research

For this paper, the algebraic equation (23) sufficed for an approximation, but alternate ways to calculate $\mathcal{W}$ include an analytical path. This could provide an exact solution, rather than an approximation. Even if a solution is not found, the attempt to further simplify and analyse the equation could give an equation that is algorithmically simpler, making it less taxing for a Mathematica program and consequently possible to approxi-
mate $W$ with higher accuracy.

As mentioned in the conclusion, it is entirely possible to insert a definition for $\mu$ or an alternate definition for $V$. This means that the Mathematica code can be used for further research within physical models such as SYM theories, taking it a step closer to "real" physics. Finally, further studies could attempt to approach the issues discussed, whether it would be increasing the efficiency of the code by rewriting it, deriving and solving an algorithmically simpler equation or changing the code so it accounts for a constant $N$. 

17
References


A Mathematica Code

The Mathematica code used to approximate $W$ can be found at https://github.com/LawanF/raysWilsonLoops/blob/main/WilsonLinSolve.nb