Computation of Wilson Loops

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Abstract

Wilson loops connects physical fields in the supersymmetric Yang-Mills Theory. The cases considered in this report contains the most symmetric field theory not including gravity, that has yet been discovered. With properties telling information about both quantum electromagnetism and quantum chromodynamics, there is and will be a great area of use for Wilson loops. Therefore, a possibility to compute them is key and is the aim of this study. In conclusion this is done by solving an integro-differential equation in terms of $\mathbb{W}(2\pi)$ and Wolfram Mathemtica. The result of this is constants making it able to compute a Wilson loop.
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1 Introduction

Before the twentieth century, physics was mainly carried by the fundamental laws consisting of Newtonian mechanics, and Maxwell’s theory of electromagnetism. However, better methods and materials resulted in unexpected observations where the macroscopic laws, classical mechanics, could not be applied to microscopic events [1]. By this time Heisenberg, Schrödinger, Dirac and many other physicists had started to formulate a new concept of physical fields called quantum theory [2]. This was revolutionary as it could be used as an important tool when describing events at the lowest of levels. It is also constantly developing and has over the years expanded to include both the field concept and the theory of relativity, which all taken together form quantum field theory [3].

1.1 Quantum chromodynamics

The main focus in finding explanations to the quantum behaviour is found in the elementary particles, organised by the Standard Model in Figure 1.

![Figure 1: The standard model containing the elementary particles [4].](image)

Electric and magnetic interactions between elementary particles are described by quantum electromagnetism. Moreover the strong force deciding the state of the parti-
cles is related to the theory of quantum chromodynamics [5]. This model was discovered in the 60s by Guell Man, Zweig and Feynman and its method is classifying charges by the quantum colors: red, blue, and green [6]. The total charge can be told by which mix it is and if all colors is of the same proportions they cancel each other out, creating white and the particle can be detected as uncharged. For example the proton consists of two up-quarks being red and blue and one down-quark being green, together they create white, which tells us that the proton does not have an electromagnetic charge [7]. Furthermore the interactions between the elementary particles is described only with predictions about the existence of the force-carrying particle, gluon, in quantum chromodynamics [8].

1.2 Møller scattering

Researchers have, against their intuition, found that particles can interact without touching, as a result of them behaving like waves. This contradiction was tested by the danish physicist Christian Møller in the early 1930s, by combining Maxwell’s equations about charge and densities and Dirac’s equations about the electron’s state transition when going from an initial to a final free state [9]. The change in state refers to different kinds of radiating emissions [10]. The experiment is called Møller scattering and is done by projecting two beams of electrons with a slight tint towards each others directions, as in Figure 2 [11].

Figure 2: Electrons $e^-$ interacting and communicating by exciting a photon $\gamma$ [12].
1.3 Yang-Mills Theory

Møller scattering can be described by a unified gauge theory called Yang-Mills Theory. Gauge theory is a mathematical theory in quantum field theory, that unifies quantum mechanics with Einstein’s special theory of relativity. It also contains an important group of transformations in the field variables, that act in a particular gauge theory and field, which gives general restrictions about interactions with other fields and elementary particles [13]. The aim of the Yang-Mills theory is to unify electrodynamics and quantum chromodynamics, that separately describes the weak and strong fundamental forces by photons and gluons [14] [15] [16]. The theory also form the basis of our understanding of the Standard Model. Furthermore Yang-Mills theory unifies the fields by using mathematical models operating through supersymmetry, meaning a symmetry between fermions and bosons [17].

However the Yang-Mills Theory depends on a number of variables, where in the simplest of forms it solely depends on the particle’s charge $e$ and the number of particles $N$. For example, in the Møller scattering, $N = 1$ as a result of the electron being the only particle containing a charge [18].

1.3.1 Extended supersymmetry

In this report two cases of the supersymmetric Yang-Mills Theory will be researched, that together can be found in Extended Supersymmetry. The physical theory, supersymmetry, is an extension of the standard model (see Figure 1) that predicts an existence for a corresponding particle to each particle described in the standard model. An advantage with the supersymmetry is that, if true, it would solve major problems with the current model, for example the mass of the Higgs boson. Therefore supersymmetry plays a great part in quantum physics and is relevant to understand [17].

The forms of the Yang-Mills theory that is relevant in this report is firstly when $\mathcal{N} = 2^*$. This means that the number of supersymmetries is equal to two. There is not as much research done in this field as in the second one, when $\mathcal{N} = 4$. In this case we have
a description of the universe in terms of different quark-fields that are related by four supersymmetries. This case is one of the simplest of Yang-Mills supersymmetries and one of the most important ones since it is a finite quantum field theory in four dimensions. Some researchers even say it is the most symmetric field theory that does not involve gravity.

1.4 Aharonov-Bohm experiment

An experimental way of showing the interactions between the elementary particles is through the Aharonov-Bohm experiment. This was introduced by a paper, called "Significance of Electromagnetic Potentials", in the 50s, that predicted an effect of magnetic fields limited from affected charged particles [19].

To begin with a solenoid (metal-coil) is placed between a plate and a screen. The plate has double slits where the electrons from $a$ can go through. The electrons go past the solenoid and can be detected when touching the screen $b$. The experiment is mainly looking at whether the pattern of light on the screen changes, when an electric current is turned on compared to off, in the solenoid, that creates a magnetic field. The detection and experiment can be seen in Figure 3.

![Figure 3: Aharonov-Bohm experiment with electrons from $a$ to $b$ and a magnetic field on ($B \neq 0$) or off ($B = 0$).](image)

The different patterns depending on the magnetic field was different from the intuition
since the electrons did not go through the solenoid and therefore should not get affected, but did get affected. However this confirms another theory, telling us that electrons behave rather like waves than particles and that we should consider both electronic and magnetic fields when analysing interactions between elementary particles.

1.5 Wilson loop

The Aharonov-Bohm experiment can be further explained using Wilson loops and quantum physics since classical physics would say that an interaction is impossible without direct contact between the electrons. The Wilson loop is drawn around the solenoid and quantifies the phase-shift between two electronic waves. These gives a better approximation of the situation since the electrons, as a result of the experiment, should be considered waves instead of particles. The Wilson loop also tells how much the interference lines move on the screen.

Since Wilson loops were first described in the 1910s they have been used to describe the strong fundamental force but today they are more commonly used to examine static points in the electric field and to identify whether particles are able to exist (or not) within bound states [20]. The bound state is a quantum state where the probability mass of the wave is concentrated in one or more regions of space. The different states are called confined and deconfined phases.

Wilson loops are also used to derive the potential between static electric charges \( q \) and \( -q \) at a distance \( r \). In classical physics it is known from Maxwell’s electromagnetism that Coloumb potential can be described with \( V(r) \propto \frac{q^2}{r} \) and Coloumb force \( F(r) \propto \frac{q^2}{r^2} \). This can be applied in classical cases, but to study when and how quarks bind to form nuclear matter it is necessary to compute \( V(r) \), which can be done with Wilson loops.

1.5.1 How to compute a Wilson Loop

The Wilson loop can be computed by using supersymmetries that allow computation of circular loops at finite coupling in the form of an integral over an \( N \times N \) matrix. The
loop acts in four dimensions (three spatial and one time: \(X = (t, x_1, x_2, x_3)\)). The loop \(\mathcal{W}\) can generally be described with the following expression:

\[
\mathcal{W}(x) \sim exp \left( \int_{\mu=0}^{3} A_{\mu} \frac{dx_{\mu}}{d\tau} d\tau \right)
\]

(1)

where \(A_{\mu}\) contain the information about the electric charges and therefore takes an important place in the equation [21]. To begin with, the description of \(A_{\mu}\) originates from Maxwell’s equation expressing \(\vec{B}\) and \(\vec{E}\), which can be written in gradient form as:

\[
\vec{E} = -\nabla \cdot V
\]

\[
\vec{B} = \nabla \cdot \vec{A}.
\]

This helps us in describing the term \(A_{\mu} = (A_0, A_1, A_2, A_3)\) where \(A_0\) is found in \(V\) and \(A_1, A_2, A_3\) is found by \(\vec{A}\). As a summary \(A_{\mu}\) contains information on both \(\vec{E}\) and \(\vec{A}\) in an electromagnetic system and therefore an explanation of a Wilson loop can give information for quantum mechanical electromagnetism.

The following represents a matrix model to describe a Wilson loop and will later be used and get calculated as a multidimensional integral

\[
\mathcal{W}(x) = \frac{1}{N} \left\langle \sum_i e^{x_{a_i}} \right\rangle.
\]

(2)

[22].

1.6 The aim of this study

The Wilson loop can be used in many areas of physics and is therefore highly relevant. When finding the value of \(\mathcal{W}(2\pi)\) we can compute a Wilson loop and therefore make multiple physical properties in quantum physics available [22]. How the procedure is done when computing the matrix in different Yang-Mills theories is therefore the aim of
2 Calculations

In this part we will find an expression for $W(x)$ to be able to compute a Wilson loop $W(2\pi)$. For that reason, the goal is primarily to construct an equation containing integrals and derivatives of $W$, since integro-differential equations easily can be solved by code in Wolfram Mathematica. Different cases of the supersymmetric Yang-Mills Theory, when $N = 2$ and $N = 4$, will finally be considered and computed with the help of a program in Wolfram Mathematica.

2.1 Background information equations

The matrix model we will study can be seen in equation (3) and is an equation over an $N \times N$ matrix.

$$W(x) = \frac{1}{N} \langle \sum_i e^{x a_i} \rangle$$

The bracket operator $\langle \ldots \rangle$ is defined as

$$\langle \ldots \rangle = \frac{1}{Z} \int \left[ \prod_i da_i \right] \langle \ldots \rangle \exp \left( -N \sum_i V(a_i) \right) \left[ \prod_{i<j} \mu(a_i - a_j) \right]$$

where

$$Z = \int \left[ \prod_i da_i \right] \exp \left( -N \sum_i V(a_i) \right) \left[ \prod_{i<j} \mu(a_i - a_j) \right].$$

Also note that

$$\int \left[ \prod_i da_i \right]$$

is an integral from $\infty$ to $-\infty$ over all $a_i$ and $\exp(x)$. In equation (4), $V$ is defined as

$$V(z) = \frac{8\pi^2}{\lambda} \sum_{m=0}^\infty T_m z^m$$

7
where $T_m$’s are constants in the Taylor expansion.

2.2 Deriving the integro-differential equation for $W$

One way to compute $W(x)$ would be from the definition (3). However this is inefficient and equations containing derivatives and integrals of $W(x)$ will be used instead to construct a integro-differential equation, that can be solved with mathematica code.

Firstly we can confirm that

$$\lim_{x \to \infty} W(x) = 0$$

$$\lim_{x \to -\infty} W(x) = 0.$$

Since

$$\int_{-\infty}^{\infty} f'(x)dx = \lim_{A \to \infty} \int_{-A}^{A} f'(x)$$

$$= \lim_{A \to \infty} [f(A)]_{-A}^{A}$$

$$= \lim_{A \to \infty} f(A) - \lim_{B \to \infty} f(-B) = 0$$

we get the equality

$$0 = \frac{1}{Z} \int \prod_i da_i \sum_p \frac{\partial}{\partial a_p} \left[ e^{x a_p} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \right].$$

The derivative $\frac{\partial}{\partial a_p}$ is a partial derivative meaning that it is deriving with respect to one of the variables $a_p$ in a multi variable function and holds the other constants.

2.2.1 Partial derivatives

To obtain an expression in terms of $W(x)$ we have to compute the partial derivative. The rule, given $Z = f(x, y)g(x, y)$

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial x} \cdot g(x, y) + \frac{\partial g}{\partial x} \cdot f(x, y)$$
\[
\frac{\partial Z}{\partial y} = \frac{\partial f}{\partial y} \cdot g(x, y) + \frac{\partial g}{\partial y} \cdot f(x, y)
\]

implies that

\[
0 = \frac{1}{Z} \int \prod_i da_i \sum_p \frac{\partial}{\partial a_p} \left[ e^{xap} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) \right]
\]

\[
= \frac{1}{Z} \int \prod_i da_i \sum_p \frac{\partial}{\partial a_p} \left( e^{xap} \right) \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) +
\]

\[
\frac{1}{Z} \int \prod_i da_i \sum_p e^{xap} \frac{\partial}{\partial a_p} \left( \exp \left( -N \sum_i V(a_i) \right) \right) \prod_{i<j} \mu(a_i - a_j) +
\]

\[
\frac{1}{Z} \int \prod_i da_i \sum_p e^{xap} \exp \left( -N \sum_i V(a_i) \right) \frac{\partial}{\partial a_p} \left( \prod_{i<j} \mu(a_i - a_j) \right).
\]

Now each derivative can be calculated separately. Firstly we have

\[
\frac{\partial}{\partial a_p} \left( e^{xap} \right) = xe^{xap}.
\]

Secondly, by applying basic rules of derivation we get:

\[
\frac{\partial}{\partial a_p} \left( \exp \left( -N \sum_i V(a_i) \right) \right) = -NV'(a_p) \prod_i \exp(-NV(a_i))
\]

\[
= -NV'(a_p) \exp \left( -N \sum_i V(a_i) \right).
\]

The third derivation is a bit more complex and we have to use the logarithmic derivative of a function \( f \):

\[
(ln f)' = \frac{f'}{f} \iff f' = f \cdot (ln f)'.
\]

Using this we get
\[
\frac{\partial}{\partial a_p} \left( \prod_{i<j} \mu (a_i - a_j) \right)
\]

\[
= \left( \prod_{i<j} \mu (a_i - a_j) \right) \frac{\partial}{\partial a_p} \ln \left( \prod_{i<j} \mu (a_i - a_j) \right)
\]

\[
= \left( \prod_{i<j} \mu (a_i - a_j) \right) \frac{\partial}{\partial a_p} \sum_{i<j} \ln (\mu (a_i - a_j))
\]

\[
= \left( \prod_{i<j} \mu (a_i - a_j) \right) \frac{\partial}{\partial a_p} \left( \sum_{i<p} \ln (\mu (a_p - a_i)) + \sum_{j>p} \ln (\mu (a_j - a_p)) \right).
\]

\[\mu\] is an even function, meaning that \(\mu(a_p - a_i) = \mu(a_i - a_p)\), and therefore we can simplify in the following way:

\[
\left( \prod_{i<j} \mu (a_i - a_j) \right) \frac{\partial}{\partial a_p} \left( \sum_{i\neq p} \ln (\mu (a_p - a_i)) \right).
\]

Using the chain rule \(\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx}\) where \(g(x) = \mu (a_i - a_j)\) and \(f(g(x)) = \ln g(x)\) we get:

\[g'(x) = \mu(a_i - a_j) = \mu'(a_i - a_j)\]

\[f'(g(x)) = \frac{1}{g(x)} = \frac{1}{\mu (a_i - a_j)}.
\]

This gives us

\[
\frac{\partial}{\partial a_p} \left( \prod_{i<j} \mu (a_i - a_j) \right) = \left( \prod_{i<j} \mu (a_i - a_j) \right) \left( \sum_{i\neq p} \frac{\mu'(a_p - a_i)}{\mu (a_p - a_i)} \right).
\]

(8)

Combining the derivatives in the equations (6), (7) and (8), we receive:

\[
0 = \frac{1}{Z} \int \prod_i da_i \sum_p \frac{\partial}{\partial a_p} \left[ e^{x a_p} \exp \left( -N \sum_i V (a_i) \right) \prod_{i<j} \mu (a_i - a_j) \right]
\]
\[
\frac{1}{Z} \int \prod_i da_i \sum_p e^{xap} \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) + \\
\frac{1}{Z} \int \prod_i da_i \sum_p e^{xap} - NV'(a_p) \exp \left( -N \sum_i V(a_i) \right) \prod_{i<j} \mu(a_i - a_j) + \\
\frac{1}{Z} \int \prod_i da_i \sum_p e^{xap} \exp \left( -N \sum_i V(a_i) \right) \left( \prod_{i<j} \mu(a_i - a_j) \right) \left( \sum_{i \neq p} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \right).
\]

which with the help of the definition of angled brackets in equation (4) gives

\[
0 = \left< x \sum_p e^{xap} \right> - N \left< \sum_p V'(a_p) e^{xap} \right> + \left< \sum_p e^{xap} \frac{\mu'(a_p - a_i)}{\mu(a_p - a_i)} \right>.
\]

2.2.2 Rewriting integro-differential equation in terms of $W(x)$

Recall the definition of $W(x)$ where we have

\[
W(x) = \frac{1}{N} \left< \sum_i e^{xai} \right>.
\]

Every term in equation (9) can be rewritten with the help of this definition. We rewrite each term separately and firstly we have:

\[
\left< \sum_p xe^{xap} \right> = NxW(x).
\]

For the second term, we have to use the definition of $V$ from equation (5):

\[
V(z) = \frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m z^m
\]

\[
V'(z) = \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} mT_m z^{m-1}.
\]
Inserting this into the second term, we get

\[-N \left\langle \sum_p V' (a_p) e^{xa_p} \right\rangle = -N \left\langle \sum_p \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} mT_m a_p^{m-1} e^{xa_p} \right\rangle = \left\langle \sum_p \frac{8\pi^2}{\lambda} \sum_{m=1}^{\infty} mT_m \left( \frac{d}{dx} \right)^m e^{xa_p} \right\rangle = -N \left\langle \sum_p V' \left( \frac{d}{dx} \right) e^{xa_p} \right\rangle = -N^2 V' \left( \frac{d}{dx} \right) \mathbb{W}(x). \tag{11} \]

For the third term, we will use the given definition

\[\gamma(x) = x \frac{\mu'(x)}{\mu(x)} \]

and by this, the last term in equation (9) can be rewritten as follows

\[\left\langle \sum_{i \neq p} e^{xa_p} \mu'(a_p - a_i) \right\rangle = \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{xa_p}}{a_p - a_i} \right\rangle = \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{xa_p} + e^{xa_i}}{2} + \frac{e^{xa_p} - e^{xa_i}}{2} = 1 \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{xa_p} - e^{xa_i}}{a_p - a_i} \right\rangle. \tag{12} \]

To further simplify this equation in terms of \( \mathbb{W}(x) \) we can use an expression for the integral

\[\int_0^x ds e^{s(a_p + (x-s)a_i)} \]
and make the substitution:

\[
\int_0^x ds e^{s(a_p - a_i)} = e^{xa_i} \int_0^x ds e^{s(a_p - a_i)} = e^{xa_i} \left[ \frac{e^{s(a_p - a_i)}}{a_p - a_i} \right]_0^x = e^{xa_i} \frac{e^{x(a_p - a_i)} - 1}{a_p - a_i} = \frac{e^{xa_p} - e^{xa_i}}{a_p - a_i}.
\]

Now we are able to rewrite equation (12) as

\[
\frac{1}{2} \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \frac{e^{xa_p} - e^{xa_i}}{a_p - a_i} \right\rangle
\]

\[
= \frac{1}{2} \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \int_0^x ds e^{s(a_p + (x-s)a_i)} \right\rangle
\]

Furthermore, we insert the Fourier transform of \( \gamma(a_p - a_i) \), that is

\[
\gamma(a_p - a_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(a_p - a_i)} \tilde{\gamma}(\omega)
\]

where

\[
\tilde{\gamma}(\omega) = \int_{-\infty}^{\infty} dx e^{i\omega x} \gamma(x).
\]

Consequently we can rewrite the equation (13) in the following way

\[
\frac{1}{2} \left\langle \sum_{i \neq p} \gamma(a_p - a_i) \int_0^x ds e^{s(a_p + (x-s)a_i)} \right\rangle
\]
\[
\begin{align*}
&= \frac{1}{2} \left\langle \sum_{i \neq p} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\omega(a_p - a_i)} \tilde{\gamma}(\omega) \int_0^x ds e^{s(a_p + (x-s)a_i)} \right\rangle \\
&= \frac{1}{2} \left\langle \sum_{i \neq p} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(a_p - a_i)} \tilde{\gamma}(\omega) \int_0^x ds e^{s(a_p + xa - sa_i)} \right\rangle \\
&= \frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_0^x ds e^{s(a_p - a_i)} e^{-i\omega(a_p - a_i)} \right\rangle \\
&= \frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_0^x ds e^{s(a_p - a_i)} e^{-i\omega(a_p - a_i)} \right\rangle - \frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \right\rangle.
\end{align*}
\]

By definition it is given that \( \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) = \gamma(0) = 2 \) which makes it possible to rewrite the term as

\[
\frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_0^x ds e^{s(a_p - a_i)} e^{-i\omega(a_p - a_i)} \right\rangle - \frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \right\rangle
= \frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_0^x ds e^{s(a_p - a_i)} e^{-i\omega(a_p - a_i)} \right\rangle - \frac{1}{2} \left\langle \sum_{i \neq p} e^{xa_i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \right\rangle
= \frac{1}{2} \left\langle \sum_{i \neq p} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_0^x ds e^{a_i(x-s+i\omega)} e^{a_p(s-i\omega)} \right\rangle - N_x \mathbb{W}(x).
\]

By definition it is known that \( \mathbb{W} \) taking two arguments is defined as follows

\[
\mathbb{W}(x, y) = \left\langle \sum_{i,j} e^{xa_i} e^{yaj} \right\rangle - N^2 \mathbb{W}(x) \mathbb{W}(x),
\]

which we can use as a substitute in equation (14) like

\[
\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int ds \left[ \mathbb{W}(x - s + i\omega, s - i\omega) + N^2 \mathbb{W}(x - s + i\omega) \mathbb{W}(s - i\omega) \right] - N_x \mathbb{W}(x).
\]

(15)
Putting all three terms (9), (11) and (15) together, we get

\[ 0 = N x \mathbb{W}(x) - N^2 V'(\frac{d}{dx}) \mathbb{W}(x) + \]

\[ + \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int ds \left[ \mathbb{W}(x - s + i\omega, s - i\omega) + N^2 \mathbb{W}(x - s + i\omega) \mathbb{W}(s - i\omega) \right] - N x \mathbb{W}(x). \]

This can be expressed by using the definition (10) for \( V'(z) \) which will give the following equation:

\[ N^2 \frac{8\pi^2}{\lambda} \sum_{p=1}^{\infty} p T_p \left( \frac{d}{dx} \right)^{p-1} \mathbb{W}(x) \]

\[ = -N^2 V'(\frac{d}{dx}) \mathbb{W}(x) + \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds [\mathbb{W}(s - i\omega, x - s + i\omega) + N^2 \mathbb{W}(s - i\omega) \mathbb{W}(x - s + i\omega)]. \]

### 2.3 Rewriting the equation in terms of \( \mathbb{W}(x) \)

From the latest section we got the following equality:

\[ \sum_{p=1}^{\infty} p T_p \left( \frac{d}{dx} \right)^{p-1} \mathbb{W}(x) \]

\[ = \frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int_{0}^{x} ds [\mathbb{W}(s - i\omega, x - s + i\omega) + N^2 \mathbb{W}(s - i\omega) \mathbb{W}(x - s + i\omega)]. \]

To continue, the following ansatz

\[ \mathbb{W}(x) = e^{ax} \tilde{\mathbb{W}}(x). \]

\[ \mathbb{W}(x, y) = e^{a(x+y)} \tilde{\mathbb{W}}(x, y) \]

will be made, where the function \( \tilde{\mathbb{W}}(x) \) is defined by

\[ \tilde{\mathbb{W}}(x) = \tilde{c}_{000} + \sum_{k=0}^{\infty} \sum_{n=2k}^{\infty} \sum_{m=1}^{2n} \tilde{c}_{k,n,m} N^{-2k} \lambda^n x^m \]
and the function $\tilde{W}(x, y)$ is defined by

$$
\tilde{W}(x, y) = \sum_{k=0}^{\infty} \sum_{n=2k+1}^{\infty} \sum_{m=1}^{2n-2m} \sum_{m'=1}^{2n-1} \tilde{d}_{k,n,m,m'} N^{-2k} \lambda^{n} x^{m} y^{m'}.
$$

(19)

Both (18) and (19) are Taylor expansions with the purpose to approximate a solution to the integro-differential equation.

2.3.1 Substitute equation with the help of $\tilde{W}(x)$

Inserting the ansatz will make a substitution for (16) possible, which will obtain an equation for $\tilde{W}(x)$ to later find an expression for $W(x)$. To begin with we have restrictions that $N \to \infty$ and $\lambda \to 0$.

Firstly we have to substitute $W(x)$ in equation (16) with help of equation (17), which gives us:

$$
\sum_{p=1}^{\infty} p T_p \left( \frac{d}{dx} \right)^{p-1} e^{a x} \tilde{W}(x) = \frac{\lambda}{16 \pi^2} \int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} \gamma(\omega) \int_{0}^{x} ds \left[ \frac{1}{N^2} W(s - i \omega, x - s + i \omega) + W(s - i \omega) W(x - s + i \omega) \right].
$$

(20)

The left hand side follows a pattern that can be described with the binomial theorem:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (n) a^0 b^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + ... + \binom{n}{n} a^0 b^n. \quad (21)$$

Expressing our series with the theorem gets us the following equation:

$$
\sum_{i=0}^{p-1} \binom{p-1}{i} \bar{a} e^{a x} \left( \frac{d}{dx} \right)^{p-1-i} \tilde{W}(x) = e^{a x} \cdot \sum_{i=0}^{p} \binom{p}{i} \bar{a} \left( \frac{d}{dx} \right)^{p-1-i} \tilde{W}(x) = e^{a x} \cdot \left( \bar{a} + \frac{d}{dx} \right)^{p} \tilde{W}(x).
$$
Therefore equation (20) can be written as:

\[
\sum_{p=1}^{\infty} p T_p e^{\bar{a} x} \left( \bar{a} + \frac{d}{d x} \right)^p \tilde{W}(x) \]

\[
= \frac{\lambda}{16 \pi^2} \int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} \gamma(\omega) \int_{0}^{x} ds \left[ \frac{1}{N^2} \tilde{W}(s - i \omega, x - s + i \omega) + \tilde{W}(s - i \omega) \tilde{W}(x - s + i \omega) \right].
\]

(22)

Which gives the proportionality:

\[ e^{\bar{a} x} \sum_{p=1}^{\infty} p T_p \left( \bar{a} + \frac{d}{d x} \right)^p \tilde{W}(x) \propto e^{\bar{a} x} V'(x) \left( \bar{a} + \frac{d}{d x} \right) \tilde{W}(x). \]

Find an \( \bar{a} \) so that all terms not containing \( \frac{d}{d x} \) disappear. Terms without \( \frac{d}{d x} \) are given when \( i = p \) in the second sum, giving the condition

\[
\sum_{p=1}^{\infty} p T_p \bar{a}^p e^{\bar{a} x} \tilde{W}(x) = 0.
\]

Since \( e^{\bar{a} x} \neq 0 \) and \( \tilde{W}(x) \) is not zero for all \( x \), we get another condition

\[
\sum_{p=1}^{\infty} p T_p \bar{a}^p = 0.
\]

Assuming \( \bar{a} \neq 0 \), we can simplify further:

\[
\sum_{p=1}^{\infty} p T_p \bar{a}^{p-1} = 0.
\]

(23)

Because of the definition of \( V'(x) \)

\[ V'(x) = \sum_{p=0}^{\infty} \tilde{t}_p \bar{a}^p, \]
we can write the equation (23) as a the derivative of $V$:

$$\sum_{p=1}^{\infty} pT_p x^{p-1} = V'(\bar{a}) = 0. \quad (24)$$

As a result of the derivative in equation (24) we know that one of the stationary points in $V(x)$ will be $x = \bar{a}$. Proceeding from this we have a new function $\tilde{V}(x)$, which we will define from $V(x)$ and that will translate so that $\tilde{V}(0) = V(\bar{a})$. Therefore the definition will be:

$$\tilde{V}(x) \equiv V(x + \bar{a}).$$

The function $V(x)$ equals the following equation:

$$\frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m x^m.$$

Which implies:

$$\tilde{V}(x) = V(x + \bar{a}) = \frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m (x + \bar{a})^m.$$

By using the binomial theorem (21) we get:

$$V(x + \bar{a}) = \frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m \sum_{n=0}^{m} \binom{m}{n} x^n \bar{a}^{m-n}.$$ 

We can then modify a new $T$ called $\tilde{T}_p$, that is defined by the following equality:

$$\frac{8\pi^2}{\lambda} \sum_{p=0}^{\infty} \tilde{T}_p x^p = \frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m \sum_{n=0}^{m} \binom{m}{n} x^n \bar{a}^{m-n}$$

$$\sum_{p=0}^{\infty} \tilde{T}_p x^p = \sum_{n=0}^{\infty} x^n \sum_{m=n}^{\infty} T_m \binom{m}{n} \bar{a}^{m-n}.$$
Note that \( n \) and \( p \) are both dummy variables and therefore only takes values for 0 and 1. We can now write an expression for \( \tilde{T}_p \).

\[
\tilde{T}_p = \sum_{m=p}^{\infty} \binom{m}{p} T_m \tilde{a}^{m-p}.
\]

Now we can express \( \tilde{V}(x) \) in terms of \( \tilde{T} \) instead:

\[
\tilde{V}(x) = \frac{8\pi^2}{\lambda} \sum_{m=0}^{\infty} T_m (x + \tilde{a})^m = \frac{8\pi^2}{\lambda} \sum_{p=0}^{\infty} \tilde{T}_p x^p.
\]

Using this, we can rewrite the original equation:

\[
\sum_{p=1}^{\infty} pT_p \left( \frac{d}{dx} \right)^{p-1} e^{\bar{a}x} \tilde{W}(x) = e^{\bar{a}x} \sum_{p=1}^{\infty} pT_p \sum_{i=0}^{p-1} \left( \binom{p-1}{i} \tilde{a}^i \left( \frac{d}{dx} \right)^{p-1-i} \right) \tilde{W}(x)
\]
\[
= e^{\bar{a}x} \sum_{p=1}^{\infty} pT_p \left( \tilde{a}^i + \left( \frac{d}{dx} \right)^{p-1} \right) \tilde{W}(x)
\]
\[
= e^{\bar{a}x} \frac{\lambda}{8\pi^2} V'(\tilde{a} + \frac{d}{dx}) \tilde{W}(x)
\]

that, according to equation (22), is equal to:

\[
\frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\gamma}(\omega) \int ds \left[ \tilde{W}(x - s + i\omega, s - i\omega) + N^2 \tilde{W}(x - s + i\omega) \tilde{W}(s - i\omega) \right].
\]

Dividing by common factors gives us:

\[
e^{\bar{a}x} V'(\tilde{a} + \frac{d}{dx}) \tilde{W}(x)
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\gamma}(\omega) \int ds \left[ \tilde{W}(x - s + i\omega, s - i\omega) + N^2 \tilde{W}(x - s + i\omega) \tilde{W}(s - i\omega) \right]
\]

and substituting using \( \tilde{W}(x) \) result in

\[
e^{\bar{a}x} V' \left( \tilde{a} + \frac{d}{dx} \right) \tilde{W}(x)
\]
\[
\begin{align*}
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int ds [e^{\tilde{a}(x-s+i\omega+s-\omega)\tilde{W}(x-s+i\omega,s-i\omega)} + N^2 e^{\tilde{a}(x-s+i\omega)\tilde{W}(x-s+i\omega)}e^{\tilde{a}(s-i\omega)\tilde{W}(s-i\omega)}] \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int ds \left[ e^{\tilde{a}x} \tilde{W}(x-s+i\omega,s-i\omega) + N^2 e^{\tilde{a}x} \tilde{W}(x-s+i\omega)\tilde{W}(s-i\omega) \right] \\
&= \frac{e^{\tilde{a}x}}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int ds \left[ \tilde{W}(x-s+i\omega,s-i\omega) + N^2 \tilde{W}(x-s+i\omega)\tilde{W}(s-i\omega) \right].
\end{align*}
\]

Dividing with \(e^{\tilde{a}x}\) on both sides, gives:

\[
V' \left( \tilde{a} + \frac{d}{dx} \right) \tilde{W}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}(\omega) \int ds \left[ \tilde{W}(x-s+i\omega,s-i\omega) + N^2 \tilde{W}(x-s+i\omega)\tilde{W}(s-i\omega) \right].
\]

### 2.4 Solvable cases

Solving the equation for \(N\) can be done by finding an expression for \(\tilde{W}(x)\), which therefore will be done in the following parts.

#### 2.4.1 Applying a Laplace-transform to integro-differential equation in terms of \(\tilde{W}(x)\)

A Laplace-transform is an integral transform converting a function of a real variable to a function of a complex variable \(s\). We will use rules of the Laplace-transformation and notate the new functions with an \(\mathcal{L}\). To begin with we have the following equation:

\[
V' \left( \frac{d}{dx} \right) \tilde{W}(x) = \int_{0}^{x} ds \tilde{W}(s)\tilde{W}(x-s). \tag{25}
\]

Given to us was that:

\[
V(z) \propto x^2 \quad \Rightarrow \quad V(x) = kx^2
\]
for some constant \( k \in \mathbb{R} \).

Therefore the derivative of \( V \left( \frac{d}{dx} \right) \) is:

\[
V' \left( \frac{d}{dx} \right) = 2k \frac{d}{dx},
\]

which makes it possible to instead write equation (25) as

\[
2k \frac{d}{dx} W(x) = \int_0^x ds W(s) W(x - s).
\]

By using standard Laplace-transformation for solving the equation’s left hand side we firstly get:

\[
\mathcal{L}\{2k \frac{d}{dx} W(x)\} = 2k \mathcal{L}\{\frac{d}{dx} W(x)\}.
\]

By definition the Laplace-transformation for a derivative \( f'(t) \) is \( sF(s) - f(0^+) \). Applying this we get

\[
= 2k (sW_L(s) - W(0^+))
\]

since \( N \to \infty \) and \( \lambda \to 0 \)

\[
\lim_{x \to 0^+} W(x) = \frac{1}{N \lambda} \int \left[ \prod_i da_i \right] \left( \sum_i e^{\alpha_i} \right) \exp \left( -N \sum_i^N V(a_i) \right) \left[ \prod_{i<j} \mu(a_i - a_j) \right] = 1
\]

and therefore equation (27) is equal to

\[
2k (sW_L(s) - 1).
\]
Using the following Laplace-transformation rule

$$\mathcal{L}\{\int_0^t f(\tau)g(t-\tau)\} = F(s) \cdot G(s)$$

for the right hand side gives:

$$\mathcal{L}\{\int_0^x ds \mathcal{W}(s) \mathcal{W}(x-s)\} = \mathcal{W}_\mathcal{L}(s) \cdot \mathcal{W}_\mathcal{L}(s).$$

Combining the left and right hand side gives the following final equation:

$$2k(s\mathcal{W}_\mathcal{L}(s) - 1) = \mathcal{W}_\mathcal{L}(s) \cdot \mathcal{W}_\mathcal{L}(s)$$

$$2ks\mathcal{W}_\mathcal{L}(s) - 2k = (\mathcal{W}_\mathcal{L}(s))^2.$$

It is now clear to see that we have an equation of the second order which we can use the quadratic formula on:

$$x^2 + px + q = 0$$

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Rewriting the equation according to the formula we get:

$$(\mathcal{W}_\mathcal{L}(s))^2 - 2ks\mathcal{W}_\mathcal{L}(s) + 2k = 0.$$  

To solve for $\mathcal{W}_\mathcal{L}(x)$ we do the following:

$$\mathcal{W}_\mathcal{L}(s) = -\frac{2ks}{2} \pm \sqrt{\left(\frac{2ks}{2}\right)^2 - 2k}$$

$$\mathcal{W}_\mathcal{L}(s) = -ks \pm \sqrt{k^2 s^2 - 2k}.$$
2.4.2 Applying Laplace-transform to the original form of $\mathbb{W}(x)$

To know whether to use the plus or minus sign we have to find another expression for $\mathbb{W}_L(s)$ and compare them. To begin with the original definition for $\mathbb{W}(x)$ is:

$$\mathbb{W}(x) = \frac{1}{N} \left\langle \sum_i e^{x_{a_i}} \right\rangle$$

and since we are only solving in terms of $x$ the Laplace tranform only regards $e^{x_{a_i}}$. This means that the following expression

$$\mathbb{W}_L(s) = \mathcal{L} \left\{ \frac{1}{N} \left\langle \sum_i e^{x_{a_i}} \right\rangle \right\}$$

instead can be written as the following

$$\mathbb{W}_L(s) = \frac{1}{N} \left\langle \sum_i \mathcal{L}\{e^{x_{a_i}}\}(s) \right\rangle.$$

The Laplace-transform rule for this case is as follows:

$$\mathcal{L}(e^{at}) = \frac{1}{s - a} (s > a).$$

Applying this in our case gives:

$$\mathbb{W}_L(s) = \frac{1}{N} \left\langle \sum_i \frac{a}{s - a_i} \right\rangle.$$

Now we have another expression for $\mathbb{W}_L(s)$ and can therefore determine whether it is plus or minus in the quadratic formula. To do this we use limits when it is plus or minus compared to our new expression, as follows:

$$\lim_{s \to \infty} -ks + \sqrt{k^2s^2 - 2k} = 0 \quad (28)$$

23
\[
\lim_{s \to \infty} -ks - \sqrt{k^2 s^2 - 2k} = -\infty
\]

and general solution:

\[
\lim_{s \to \infty} \frac{1}{N} \left( \sum_{i} \frac{a}{s - a_i} \right) = 0.
\] (29)

Since the equation (28) and equation (29) cohere to a greater extent, we can justify that we choose plus instead of minus. Our final expression for \( \mathbb{W}_L(x) \) is

\[
\mathbb{W}_L(s) = -ks + \sqrt{k^2 s^2 - 2k}.
\]

This definition we can use to find an expression for \( \mathbb{W} \) with the help of Wolfram Mathematica and the following code:

\[
\text{InverseLaplaceTransform}[-s + \sqrt{s^2 - a}, \{a\}, \{x\}].
\]

Our result for \( \mathbb{W}(x) \) is the following equation:

\[
-\frac{\sqrt{a}\text{BesselI}[1, \sqrt{ax}]}{x} + \text{Diracdelta}[x]
\]

which can be simplified to the following equation (\( \text{Diracdelta}[x] \approx 0 \)):

\[
-\frac{\sqrt{a}\text{BesselI}[1, \sqrt{ax}]}{x} = -\frac{\sqrt{a}\text{I}_1[1, \sqrt{ax}]}{x}.
\]

2.4.3 Calculating Wilson loop in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills Theory

The equation we get is specifically for the value of the circular Wilson loop in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills Theory. This can be confirmed since we in this case put \( T_2 = 1 \)
and $x = 2\pi$ in the original function:

$$W(x) = \frac{4\pi\sqrt{T_2}}{x\sqrt{\lambda}}I_1\left(\frac{x\sqrt{\lambda}}{2\pi\sqrt{T_2}}\right).$$

As the following:

$$W(2\pi) = \frac{4\pi\sqrt{T}}{2\pi\sqrt{\lambda}}I_1\left(\frac{2\pi\sqrt{\lambda}}{2\pi\sqrt{T}}\right) = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda}).$$

### 2.4.4 Calculating Wilson loop in $\mathcal{N} = 2^*$ supersymmetric Yang-Mills Theory

To calculate the Wilson loop $W(2\pi)$ in $\mathcal{N} = 2^*$ we have to find an expression for $\gamma$ and its derivatives (it will later become clear why) and therefore we have to consider the following expression:

$$\mu(x) = x^2 \frac{H^2(x)}{H(x - m)H(x + m)}$$

with

$$H(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right)^n e^{-\frac{z^2}{\pi}}.$$

We begin by finding an expression in terms of $H$ and $x$, for:

$$\gamma(x) = x \frac{d}{dx} (\log \mu(x)) = ...$$

According to basic derivation rules we know that:

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

which implies the following for a function $f(x)$:

$$\frac{d}{dx}(\log_a f(x)) = \frac{1}{f(x) \ln a} \cdot f'(x).$$
This applied to our equation gives:

\[
\gamma(x) = x \frac{d}{dx} (\log \mu(x)) = x \cdot \frac{1}{\mu(x) \ln a} \cdot \mu'(x).
\]

Since all calculations have been done in base \(e\) then \(a = e\) which makes \(\ln a = 1\), the equation can be simplified to:

\[
\gamma(x) = x \cdot \frac{1}{\mu(x)} \cdot \mu'(x).
\]

Using the given definition for \(\mu(x)\), that is

\[
\mu(x) = \frac{x^2 \cdot H^2(x)}{H(x - m)H(x + m)}.
\]

We are able to express \(\gamma(x)\) as

\[
x \frac{d}{dx} \log \mu(x) = 2 + x \left(2 \frac{H'(x)}{H(x)} - \frac{H'(x - m)}{H(x - M)} - \frac{H'(x + m)}{H(x - m)}\right) \tag{30}
\]

(see appendix for detailed calculations). We were given the following equality by definition:

\[
2 \frac{H'(x)}{H(x)} - \frac{H'(x - m)}{H(x - M)} - \frac{H'(x + m)}{H(x - m)} = 4 \int_0^\infty d\omega \frac{\sin^2 M \omega \sin 2\omega z}{\sinh^2 \omega}
\]

which makes it able to express equation (30) in terms of an integral in the following way

\[
\gamma(x) = x \frac{d}{dx} \log \mu(x) = 2 + 4x \left(\int_0^\infty d\omega \frac{\sin^2 M \omega \sin 2\omega z}{\sinh^2 \omega}\right). \tag{31}
\]

To continue we have to take the \(n\)th derivative of \(\gamma(0)\) and another equation that was given and looks as follows

\[
2\delta_{n,0} + 4 \int_0^\infty d\tilde{\omega} \sin^2 (M \tilde{\omega}) \left[\sum_{m=1}^\infty 4m e^{-2m\tilde{\omega}}\right] \left[\frac{1 + (-1)^n}{2} (-1)^n \frac{n+2}{2} n(2\tilde{\omega})^{n-1}\right]. \tag{32}
\]
where Kronecker delta \((\delta_{n-m})\) means the worth one if \(n = m\) and zero if \(n \neq m\).

We will compare the results to see if equation (31) and (32) is equal and if equation (32) therefore can replace equation (31). We can thus more easily integrate a part of the derivate in later equations. The comparing is done with a code that generates the first 15\(^{th}\) derivations of both equations.

**Mathematica Input:**

```mathematica
f[x_] := 2 + 4x Divide[Sin[M w]^2] Sin[2w x], Sinh[w]^2]
D[f[x], {x, n}]/.x->0;
Table[%, {n,0,15}]//FullSimplify

2 KroneckerDelta[0,n]+4Sin[M w]^2 Sum[4m Exp[-2m w],{m,1,\[Infinity]} ]
Divide[1+(-1)^n,2](-1)\[Gamma](2w)^n; Table [\%, \{n,0,15\}]// FullSimplify

—:
Integrate[4(Sin[M w]^2 4m Exp[-2m w](w)^n-1), \{ w,0,\[Infinity]\} ,]
GenerateConditions\[Rule]False] // Expand

As output we get the following equality:

\[
2^{3-n}m^{1-n}\[Gamma](n) - 4m(2m - 2iM)^{-n}\[Gamma](n) - 4m(2m + 2iM)^{-n}\[Gamma](n)
\]

that can be simplified to

\[
= 2^2\[Gamma](n)(2^{1-n}m^{1-n} - m(2m - 2iM)^{-n} - m(2m + 2iM)^{-n}).
\] (33)

The definition of the gamma function is

\[
\[Gamma](n) = (n - 1)!.\]

The equation (33) can be simplified using the Riemann Zeta Function, that is defined for
one variable as follows

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$$

and for two variables as

$$\zeta(s, a) = \sum_{m=1}^{\infty} \frac{1}{(m + a)^s}.$$ 

In cases where the equation looks like

$$\sum_{m=1}^{\infty} \frac{m}{(m + a)^s}$$

we rewrite it in the following way and make it possible to continue using the Riemann Zeta Function:

$$\sum_{m=1}^{\infty} \frac{m + a - a}{(m + a)^s} = \sum_{m=1}^{\infty} \frac{m + a}{(m + a)^s} - \frac{a}{(m + a)^s} = \sum_{m=1}^{\infty} \frac{1}{(m + a)^{s-1}} - \frac{a}{(m + a)^s} = \zeta(s - 1, a) - a\zeta(s, a).$$

Recall the output in equation (33), here we have three terms inside the parenthesis and also keep in mind that everything is done in a series. Therefore we can rewrite it with the method of the Riemann Zeta Function, this will be done in the separate parts according to which term:

Firstly:

$$\sum_{m=1}^{\infty} 2^{1-n}m^{1-n} = \sum_{m=1}^{\infty} \frac{2}{2^n \cdot m^{n-1}} = \frac{1}{2^n} \cdot 2 \cdot \zeta(n - 1)$$
Secondly:

\[
\sum_{m=1}^{\infty} m(2m - 2iM)^{-n} = \sum_{m=1}^{\infty} \frac{m}{(2m - 2iM)^n} = \frac{1}{2^n} \sum_{m=1}^{\infty} \frac{m - iM}{(m - iM)^2} + \frac{iM}{(m - iM)^n} = \frac{1}{2^n} \zeta(n - 1, -iM) + iM\zeta(n, -iM)
\]

Lastly:

\[
\sum_{m=1}^{\infty} m(2m + 2iM)^{-n} = \sum_{m=1}^{\infty} \frac{m}{(2m + 2iM)^n} = \frac{1}{2^n} \sum_{m=1}^{\infty} \frac{m + iM}{(m + iM)^n} + \frac{iM}{(m + iM)^n} = \frac{1}{2^n} \zeta(n - 1, iM) - iM\zeta(n, -iM)
\]

This gives one part of the left hand side of final solution:

\[
\frac{1}{2^{n-1}} \Gamma(n)(2\zeta(n - 1) - \zeta(n - 1, -iM) - iM\zeta(n, -iM) - \zeta(n - 1, iM) + iM\zeta(n, iM)).
\] (34)

Plugging in constants into the above (34) we get the whole final solution for the derivatives of \(\gamma\):

\[
\left(\frac{d}{dx}\right)^n \gamma(0) = 2\delta_{n,0} + 4 \frac{1 + (-)^n}{2} \left(-\right)^{\frac{n+2}{2}}.
\]

\[
\cdot n \left[ \frac{\Gamma(n)}{2} (2\zeta(n - 1) - \zeta(n - 1, -iM) - iM\zeta(n, -iM) - \zeta(n - 1, iM) + iM\zeta(n, iM)) \right].
\]

Inserting this to Wolfram Mathematica will give an output containing all constants needed to compute a Wilson loop in \(\mathcal{N} = 2^*\) Supersymmetric Yang-Mills Theory.
3 Results

When $\lambda = 1$ in the supersymmetric Yang-Mills theory the following constants were calculated, which purpose is to compute $\mathcal{W}(x)$ and since $x = 2\pi$ the Wilson loop $\mathcal{W}(2\pi)$ is computed.

\[
\begin{align*}
\{\tilde{c}[0, 1, 1], \tilde{c}[0, 1, 2]\} \\
\{\tilde{c}[0, 1, 1], \tilde{c}[0, 1, 2], \tilde{c}[0, 2, 1], \tilde{c}[0, 2, 2], \tilde{c}[0, 2, 3], \tilde{c}[0, 2, 4]\} \\
\{\tilde{c}[0, 1, 1], \tilde{c}[0, 1, 2], \tilde{c}[0, 1, 1] \rightarrow 0, \tilde{c}[0, 1, 2] \rightarrow 0, \\
\tilde{c}[0, 2, 1] \rightarrow -\frac{54}{1024} \tilde{y}^2 \tilde{y}^3 \tilde{y}^4 - 8 \tilde{y}^2 \tilde{y}^3 \tilde{y}^4 + \frac{18}{1024} \tilde{y}^2 \tilde{y}^3 \tilde{y}^4 \gamma[0], \tilde{c}[0, 2, 2] \rightarrow \frac{18 \tilde{y}^2 - 8 \tilde{y}^2 \tilde{y}^4 + \frac{1024}{1024} \tilde{y}^2 \tilde{y}^4 \gamma[0]}{1024 \pi^2 \tilde{y}^2}, \tilde{c}[0, 2, 3] \rightarrow -\frac{\tilde{y}^2}{1024 \pi^2 \tilde{y}^2}, \tilde{c}[0, 2, 4] \rightarrow \frac{1}{12288 \pi^2 \tilde{y}^2}\}
\end{align*}
\]

Figure 4: When $\lambda = 1$

See Appendix for case when $\lambda > 1$.

4 Discussion

In this section a conclusion, from the result, will be drawn. Opportunities for further research based on our results will also be presented.

4.1 Conclusion

In this study the Wilson loop have for two special cases been computed with an algebraic method and by using the software Wolfram Mathematica. In conclusion an integro-differential equation in the form $\mathcal{W}$ was derived. We later solved this for small values of $\lambda$ with perturbation theory. This resulted in equations containing $\tilde{\mathcal{W}}(x)$ coefficients, that mathematica solved as a linear expression. Furthermore $\tilde{\mathcal{W}}(x)$ is a Taylor series and the coefficients can therefore obtain an expression for $\mathcal{W}$, which can and did compute the general Wilson loop $\mathcal{W}(2\pi)$.
4.2 Further research

To begin with, Wilson loops, are highly relevant as a result of being one of the fundamental operators in any gauge theory. More specifically further research in this field would primarily focus on more general cases, since this paper focused on applying to some specific cases (SYM $\mathcal{N} = 4$ and SYM $\mathcal{N} = 2^*$). This is important since it could be used in many cases without having to be modified. Secondly it would be relevant to compute Wilson loops for finite $\lambda$ in the integro-differential equation, which could be solved numerically instead of through a derivative. Lastly a solution for specific $N$ would be rather useful, instead of approximating $N$ going to infinity. This is a reasonable simplification and is used to get a result in an easier way. Finally this should get corrected to be able to see a greater correlation between calculations and observations.
References


5 Appendix

5.1 Calculation of $\gamma(x)$

$$\gamma(x) = x \frac{d}{dx} (\log \mu(x))$$

$$= x \cdot \frac{1}{\mu(x) \ln a} \cdot \mu'(x)$$

Since all calculations have been done in base $e$ then $a = e$ which makes $\ln a = 1$, the equation can be simplified to:

$$\gamma(x) = x \cdot \frac{1}{\mu(x)} \cdot \mu'(x)$$

We will hereafter calculate $\mu(x)$ and $\mu'(x)$ to be able to calculate $\gamma(x)$. To begin with, let us introduce the quotient rule for

$$P(x) = \frac{f(x)}{g(x)}$$

$$P'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

and then separate $\mu(x)$ into two different functions:

$$\mu(x) = \frac{x^2 \cdot H^2(x)}{H(x - m)H(x + m)} = \frac{f(x)}{g(x)}$$

where

$$f(x) = x^2 \cdot H^2(x)$$

$$g(x) = H(x - M)H(x + M).$$

When deriving $f(x)$ and $g(x)$ we have to use the product rule for derivation:

$$P(x) = f(x) \cdot g(x) \cdot h(x)$$
\[ P'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \]

which will look like this

\[ f'(x) = 2xH^2(x) + x^2H'(x)H(x) + x^2H(x)H'(x) \]

\[ g'(x) = H'(x + M)H(x - M) + H(x + M)H'(x - M) \]

The original expression for the derivative of \( \mu(x) \):

\[ \mu'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \]

can now be written like

\[ \frac{1}{\mu(x)} = \frac{H(x - M)H(x + M)}{x^2H^2(x)} \]

gives the final expression:

\[ \mu'(x) \cdot \frac{1}{\mu(x)} = \left( \frac{2xH^2(x) + x^2(H'(x)H(x) + H(x)H'(x)))(H(x - M)H(x + M))}{(H(x - M)H(x + M))^2} \right. \]

\[ - \frac{(x^2 \cdot H^2(x))(H'(x + M)H(x - M) + H(x + M)H'(x - M))}{(H(x - M)H(x + M))^2} \cdot \frac{H(x - M)H(x + M)}{x^2H^2(x)} \]

This can by division of common factors be simplified all the way up to the following expression:

\[ x \frac{d}{dx} \log \mu(x) = 2 + x \left( \frac{2H'(x)}{H(x)} - \frac{H'(x - m)}{H(x - M)} - \frac{H'(x + m)}{H'(x - m)} \right) \]
5.2 Result when \( \lambda > 1 \)

\[
\begin{align*}
[\epsilon(0, 1, 1), \epsilon(0, 1, 2)]
\end{align*}
\]

\[
\begin{align*}
\epsilon(0, 1, 1) & \sim \frac{2}{3} \beta_1 \epsilon(0, 1, 2) + \frac{1}{64 \lambda} \beta_1^2 \epsilon(0, 1, 1) + 0, \\
\epsilon(0, 1, 2) & \sim \frac{1}{64 \lambda} \beta_1^2 \epsilon(0, 1, 1) + 0, \\
\epsilon(0, 2, 1) & \sim -54 \beta_1 + 72 \beta_2 \epsilon(0, 1, 1) - 29 \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 1) + 3 \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 2), \\
\epsilon(0, 2, 2) & \sim 18 \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 1) - 8 \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 2) + 3 \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 3) + 1 \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 4) + \frac{1}{1024} \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 5) + \frac{1}{12288} \beta_2^2 \epsilon(0, 1, 1) \epsilon(0, 2, 6), \\
\epsilon(0, 2, 3) & \sim \frac{1}{64 \lambda} \beta_1^2 \epsilon(0, 1, 1) + 0, \\
\epsilon(0, 2, 4) & \sim \frac{1}{64 \lambda} \beta_1^2 \epsilon(0, 1, 1) + 0, \\
\epsilon(0, 2, 5) & \sim \frac{1}{64 \lambda} \beta_1^2 \epsilon(0, 1, 1) + 0, \\
\epsilon(0, 2, 6) & \sim \frac{1}{64 \lambda} \beta_1^2 \epsilon(0, 1, 1) + 0.
\end{align*}
\]

Figure 5: When \( \lambda = 2 \)