Promoting Algebra Readiness: Teaching Rational Numbers to Support Student Success in Mathematics

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2019 ORTli Conference
April 26, 2019

The research reported here was supported by the Institute of Education Sciences, U.S. Department of Education, through Grant R324A120115 to the University of Oregon. The opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education.
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Agenda

- Background and rationale for Project PAR
- PAR intervention focus and scope
- Pilot study research questions and results
- Future directions and next steps
NAEP Results

• Sixty percent of 4th graders are at or below proficient

• Two areas of challenge
  – Students who struggled with whole number who lack both the conceptual understanding of fractions and the skills to work with fraction operations
  – Students who are “on-track” but fail to transition to the rational number system.
Beyond Testing Results

A game board is shown.

Some of the squares on the board are labeled.

Drag letters into the rest of the squares so that:
- \( \frac{1}{2} \) of all the squares on the board are labeled Y,
- \( \frac{1}{4} \) of all the squares on the board are labeled B, and
- \( \frac{1}{4} \) of all the squares on the board are labeled G.

Only 25\% of 4\textsuperscript{th} grade students correctly completed the problem.

Similar results from items requiring comparing and ordering fractions (e.g. 1/3 and 1/4).
Lack of Research in the Area

• Overall ratio of research in math compared to reading
• Most work has focused on whole numbers and the research base in that area is solid
• Recent meta–analyses highlighted the lack of research on fractions and algebra readiness (Shin & Bryant, 2015; Stevens et al., 2018).
  – Analyses of fraction interventions found 22 studies of varying quality and results (Hwang et al., 2018).
Background and Rationale for PAR

• Focus on fractions and rational numbers as gateway to Algebra

• Lack of intervention programs and quality research in the area

• Particular focus on students with or at-risk for math disabilities
Intervention Logic Model

**Intervention Components**

- Component: Rational Number Content
  2. Conceptual Development of Fractions
  3. Understanding of Equivalent Representations
  4. Conceptual Understanding and Procedural Fluency of Operations with Fractions

**Mediator**

- Quality of Teacher Student Interactions

**Proximal Outcome**

- Procedural Fluency

**Distal Outcome**

- Student Mathematics Achievement
  - Conceptual Understanding

**Component: Instructional Design and Delivery**

- Design Features
- Models of Math Concepts
PAR Development and Innovation

- Three-year project, using mixed methods design (Creswell, 2006; Shavelson et al., 2000)
- First two years focused on iterative development of the curriculum using design experiment methods (Brown, 1992)
- Third year used rigorous, experimental methods to conduct a quasi-experimental pilot study of the fully developed PAR intervention (WWC, 2010)
Design Process: Three Implementation Studies

**Brief Learning Trials**
- **GOALS:** Test usability and feasibility of math models, instructional strategies, and portions of lessons

**Feasibility Study**
- **GOALS:** Test usability and feasibility of FUSION in its entirety

**Pilot Study**
- **GOAL:** Target student outcomes to assess the promise of the intervention

* Followed by Efficacy Trials
PAR CURRICULUM OVERVIEW
PAR Intervention Focus & Scope

- **Tier 2 intervention** designed to be delivered to groups of 5-15 students.
- **95 lessons** divided into 4 strands typically delivered 4-5 days per week for 45 minutes per day.
- Instructional objectives **aligned with the Common Core State Standards** in Mathematics for fractions from 3rd to 6th grade.
- **Goal**: Promote algebra readiness for students at risk for mathematics difficulty.
PAR is Aligned Evidence Based Practices for Mathematics Instruction

See summary recommendations from each at end of this presentation.
PAR Intervention Strands link to CCSS-M

- **Strand 1: Multiplication and Division of Whole Numbers**
  - Review/teach important whole number concepts/skills that provide a foundation for fractions
    - Operations and Algebraic Thinking standards spanning grades 3-5
    - Number and Operations in Base Ten spanning grades 3-5.
  - Grade 6 Target Knowledge and Skills
    - 6.NS.2 – Fluently divide multi-digit numbers
    - 6.NS.4 – Work with greatest common factors and the distributive property
PAR Intervention Strands link to CCSS-M

• **Strand 2: Fractions as Numbers**
  – Review/teach important foundational fraction concepts/skills
    • Number and Operations – Fractions from Grade 3 and 4
  – Grade 6 Target Knowledge and Skills
    • 6.NS.6 – Understand a rational number as a point on the number line
PAR Intervention Strands link to CCSS-M

• **Strand 3: Addition and Subtraction with Fractions**
  – Review/teach important fraction concepts/skills related to addition and subtraction of fractions
    • Number and Operations – Fractions from Grade 4 and 5 (addition/subtraction standards)
  – Grade 5 Target Knowledge and Skills
    • 5.NF.1 – Add and subtractions with like and unlike denominators
    • 5.NF.2 – Solve word problems involving addition and subtraction of fractions
• **Strand 4: Multiplication and Division with Fractions**
  – Review/teach important fraction concepts/skills related to multiplication and division of fractions
    • Number and Operations – Fractions from Grade 4 and Grade 5 (multiplication and division standards)
  – Grade 6 Target Knowledge and Skills
    • 6.NS.1 – Interpret and compute quotients of fractions and solve word problems requiring division with fractions
Curriculum Features

Lesson Design
- Explicit teacher demonstrations
- Guided practice
- Independent
- Math models
  - Number line
  - Area models
- Student discourse

Lesson Features
- PowerPoints
- Defined precise vocabulary
- Entry tasks
- Exit tickets
- Anchor charts
- Differentiated practice opportunities
  - Extend Up
  - Extend Down

Progress Monitoring
- Performance assessments
- Strand pre/post assessments
Teacher Demonstration

Today we are going to model multiplication of two fractions that are each less than one and determine the product using the area model.

Display Activity Practice 1 and write on the equation line \( \frac{1}{3} \times \frac{1}{4} \).

Read this expression. \( \frac{1}{3} \times \frac{1}{4} \). Let’s represent \( \frac{1}{3} \) on the horizontal number line and \( \frac{1}{4} \) on the vertical number line.

Model the length and width as segments on the number line. Have students do the same on their area model. Monitor and assist as needed.

Now that we have modeled the length and width as segments, let’s partition the grid by extending each third mark from the horizontal number line across 1 unit like this.

Now partition the grid again by extending each fourth mark from the vertical number line across 1 unit like this.

How many rectangles are in this 1 sq. unit? (12.)

Insert a check mark in one rectangle to represent the area of \( \frac{1}{3} \) of a unit in length by \( \frac{1}{4} \) of a unit in width.

This rectangle represents the area of \( \frac{1}{3} \) of a unit in length by \( \frac{1}{4} \) of a unit in width. What fraction represents the rectangle with the check mark? (\( \frac{1}{12} \)).

That tells us that each rectangle equals \( \frac{1}{12} \) sq. unit and that \( \frac{1}{12} \) is the product of \( \frac{1}{3} \times \frac{1}{4} \). So, we can say that \( \frac{1}{3} \times \frac{1}{4} \) equals \( \frac{1}{12} \).

Add to the expression: \( \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \).

Partner 2, explain to your partner how the model shows that \( \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \).
ENTRY TASK ANSWER

When multiplying fractions, when will a product equal a fraction less than 1?

① When multiplying a fraction equal to 1 by a fraction less than 1.
② When multiplying a fraction equal to 1 by a fraction greater than 1.

Exit Ticket

1. Which product will be less than 1 and less than the two fraction factors?

\[
\frac{9}{10} \times \frac{7}{8} \quad \frac{1}{2} \times \frac{5}{4}
\]

2. Which product will be less than 1 and less than the two fraction factors?
Precise Vocabulary Definitions

• Mathematically correct AND student friendly

<table>
<thead>
<tr>
<th>Vocabulary (new in bold)</th>
</tr>
</thead>
</table>
| **Distributive Property of Multiplication** – The product of a number and a sum is equal to the sum of the individual products. (Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find $8 \times 7$ as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$.)
| **Variable** – A symbol for a number we do not know. It is usually a letter like $x$ or $y$.
| **Array** – Items (objects, numbers, squares, etc.) arranged in rows and columns.
| **Commutative Property of Multiplication** – Changing the order of the factors does not change the product.
| **Equation** – A mathematical statement that shows two expressions are equivalent.
| **Factors** – The numbers multiplied to get a product.
| **Multiple** – The product of any counting number and another whole number.
| **Multiplication** – Repeated addition or the joining of equal groups. Multiplication is also the inverse of division.
| **Product** – The result of multiplying two or more numbers. |
Lesson Features: Anchor Charts

**Representations of Fractions**

| PART-WHOLE |  
| EXPRESSION | \( \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \)  
| EQUIVALENT FRACTION | \( \frac{6}{8} \)  
| DECIMAL | 0.75  
| POINT ON THE NUMBER LINE |  
| DISTANCE BETWEEN TWO POINTS |  
| SUM OF EQUAL INTERVALS |  

\( \frac{3}{4} \)
Lesson Features: FOPS Problem Solving Guide

**F**ind
- Did I read and retell the problem in my own words?
- Did I determine the type of problem?
  
  **Check Problem Type:** □ Equal Groups □ Array/Area □ Compare

**O**rganize

**P**lan
- Did I write an equation using the diagram?

**S**olve
- My estimate is ____________________
  - Did I make an estimate?
  - Did I solve for the variable in the equation?
  - Did I label the answer?
  - Did I check if my answer is close to my estimate?

---


Introduced in Strand 1

Extended in Strand 3 & 4

**Jeans**

<table>
<thead>
<tr>
<th>Designer jeans cost $80. Regular jeans cost ( \frac{1}{8} ) as much as the designer jeans. How much do regular jeans cost?</th>
<th>Regular jeans cost $20 which is ( \frac{1}{5} ) of the cost of designer jeans. How much do the designer jeans cost?</th>
<th>Designer jeans cost $75 and regular jeans cost $30. What fraction of the cost of the designer jeans is the cost of the regular jeans?</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Diagram: ( 80 \times \frac{1}{8} = 10 ) dollars]</td>
<td>[Diagram: ( 80 \times \frac{1}{4} = 20 ) dollars]</td>
<td>[Diagram: ( 75 \times \frac{2}{5} = \frac{30}{1} ) or 30 dollars]</td>
</tr>
</tbody>
</table>
Lesson Features
Built-In Opportunities to Extend Up or Down

Guided Practice
Using Activity Practice 1, have students find the expression that matches the statement and write it in the space provided. Have them reason about the size of the product and write the corresponding letter in the space provided.

→ Extend Down: Provide area models as a reference for students as they complete the worksheet.

Independent Practice
Distribute Activity Practice 2 and have students determine the size of the product.

↑ Extend Up: Have students write equations that represent different product sizes.

↓ Extend Down: Have students work with a partner to discuss each problem, then have them solve the problems independently.
Progress Monitoring: Performance Tasks

Lesson 3 Performance Task 2.1

Performance Task 2.1

Name: ______________________

1. One candy bar is shared equally between 4 friends. How much of the candy bar will each student get?
   1a. Write an equation that represents how much of the candy bar each friend will get.

1b. Represent how much of the candy bar each friend will get on a number line.

1c. What fraction of the candy bar did each friend get?

2. Which number has the greatest distance from 0?
   A. $\frac{1}{3}$  B. $\frac{1}{2}$  C. $\frac{1}{4}$  D. $\frac{1}{5}$

3. Which number has the greatest distance from 0?
   A. $\frac{2}{2}$  B. $\frac{1}{3}$  C. $\frac{2}{2}$  D. $\frac{3}{1}$


Lesson 15 Performance Task 3.2

1. Which two fraction expressions are equivalent to $3 + \frac{2}{5}$?
   a. $\frac{3}{4} + \frac{2}{5}$  b. $\frac{3}{1} + \frac{2}{5}$  c. $\frac{1}{2} + \frac{2}{5}$  d. $\frac{2}{5} + \frac{2}{5}$

2. Which property should be used to write an equivalent equation of $\frac{2}{1} + \frac{2}{5}$?
   a. The commutative property of addition
   b. The associative property of addition
   c. The identity property of multiplication
   d. The distributive property

3. What fraction should be used to solve the equation, $(2 + \frac{2}{5}) - \frac{2}{5} = N$?
   a. $\times \frac{1}{5}$  b. $\times \frac{2}{5}$  c. $\times \frac{5}{10}$  d. $\times \frac{7}{2}$

4. Solve the following equation.
   4a. $(3 - \frac{1}{2}) + \frac{2}{7} = N$. Show your work below.  4b. $N = ______

5. Which one of the equations represents the word problem? (You may use the FOPS template.)
   Linda had $2\frac{3}{4}$ pounds of sugar in a container. She used some for cooking and now she has $\frac{5}{7}$ pounds of sugar left. How much sugar did she use for cooking?
   a. $A + 2\frac{3}{4} = \frac{5}{4}$  b. $A + B = 2\frac{3}{4}$
   c. $2\frac{3}{4} + \frac{5}{7} = T$  d. $\frac{5}{3} + B = 2\frac{2}{4}$
Strand 1: Multiplication and Division of Whole Numbers (25 lessons)

• Understand and Relate Multiplication & Division
  • Factors/products/quotients/multiples
    – Array and area models
    – Properties of whole numbers

• Multiplication and Division Problem Solving
  – Rounding and estimation
  – Solve equal groups, area/array, multiplicative compare problems

• Solve Equations using Order of Operations

• Multi-Digit Whole Number Multiplication and Division
  • Area models; partial products; standard algorithm
Represent and reason about multiplicative relationships using number line, area, and array models.

Number Line Model

Array Model
(draw circles, squares, or make dots)

Area Model

Answer the following questions:
1. What are the factors for each model? 3 and 7
2. What is the product? 21
3. 21 is a multiple of what two numbers? 3 and 7
4. Explain: 3 and 7 are factors of 21
5. Write an equation that represents these models. 3 x 7 = 21
Anchor Chart:
Apply properties of operations as strategies to multiply and divide whole numbers.

Number Properties of Multiplication

Zero Property
Any number multiplied by zero is zero
\(5 \times 0 = 0\)  \(\frac{3}{8} \times 0 = 0\)  \(a \times 0 = 0\)

Identity Property
A number does not change when multiplied by 1
\(a \times 1 = a\)

Commutative Property
Changing order of factors does not change product
\(a \times b = b \times a\)

Whole Numbers  | Fractions
---|---
1 | \[\begin{array}{c}
4 \\
4 \times 1 = 4
\end{array}\]

Whole Numbers  | Fractions
---|---
\[\begin{array}{c}
2 \end{array}\]  | \[\begin{array}{c}
3 \end{array}\]
| \[\begin{array}{c}
2 \end{array}\]
| \[\begin{array}{c}
3 \end{array}\]

Whole Numbers  | Fractions
---|---
\[\begin{array}{c}
1 \end{array}\]  | \[\begin{array}{c}
3 \end{array}\]
| \[\begin{array}{c}
(4 \times 1) \times 3 = 12\end{array}\]
| \[\begin{array}{c}
(3 \times 1) \times 4 = 12\end{array}\]

Whole Numbers  | Fractions
---|---
| \[\begin{array}{c}
2 \end{array}\]
| \[\begin{array}{c}
(2 \times 3) + (2 \times 5) = 16\end{array}\]

Whole Numbers  | Fractions
---|---
| \[\begin{array}{c}
2 \end{array}\]
| \[\begin{array}{c}
2 \times (3 + 5) = 16\end{array}\]

Whole Numbers  | Fractions
---|---
| \[\begin{array}{c}
1 \end{array}\]
| \[\begin{array}{c}
(3 + 5) \times 4 = 12\end{array}\]

Whole Numbers  | Fractions
---|---
| \[\begin{array}{c}
2 \end{array}\]
| \[\begin{array}{c}
(3 \times 2) + (2 \times 5) = 16\end{array}\]

Whole Numbers  | Fractions
---|---
| \[\begin{array}{c}
1 \end{array}\]
| \[\begin{array}{c}
(4 \times 1) \times 3 = 12\end{array}\]

Whole Numbers  | Fractions
---|---
| \[\begin{array}{c}
2 \end{array}\]
| \[\begin{array}{c}
(2 \times 3) + (2 \times 5) = 16\end{array}\]

Associate Property
When 3 or more factors are multiplied the product is the same regardless of the grouping of the factors
\((a \times b) \times c = a \times (b \times c)\)

Distributive Property
The product of a number and a sum is equal to the sum of the individual products
\(a \times (b + c) = (a \times b) + (a \times c)\)
Apply properties of operations as strategies to multiply and divide whole numbers
Links to Strand 4: Apply properties of operations as strategies to multiply and divide fractions

\[
\frac{2}{3} \times \frac{7}{4} + \left( \frac{2}{3} \times \frac{3}{4} \right) \left( \frac{2}{3} \times \frac{4}{4} \right) + \left( \frac{2}{3} \times \frac{3}{4} \right) \left( \frac{4}{4} \right)
\]

\[
= \frac{8}{12} + \frac{6}{12} = \frac{14}{12}
\]
Strand 2: Fractions as Numbers (15 lessons)

• Understand Fractions as Division
  – Equal sets; array/area models
  – \( a/b = a \div b \)

• Understand Unit Fractions on a Number Line
  – Unit intervals, partitioning intervals
  – Fraction as a point, distance, and distance between two points

• Iteration and Operations (conceptual)
  – Join and separate unit fractions

• Compare Two Fractions with Like Numerators or Denominators

• Understand and Generate Equivalent Fractions
Understand Fractions as Division

- Context
- Model
- Division equation
- Fraction product
Understand Unit Fractions on a Number Line

*Students partition a unit interval to represent equal intervals.*

- There are 4 equal intervals within this unit interval.
- The denominator of a fraction indicates the number of intervals within a unit interval. \( \frac{4}{4} \)
- The numerator indicates how many intervals within the unit interval are represented. \( \frac{4}{4} \)
- What unit fraction represents one of the intervals? \( \frac{1}{4} \)
Iteration and Operations (Conceptual)

Students represent iterations of a unit fraction to represent a fraction.

- Forward jumps are used to represent an addition operation.
- The expression $\frac{1}{3} + \frac{1}{3}$ represents the joining of two $\frac{1}{3}$ intervals.
- 2 intervals (or copies) of $\frac{1}{3}$ equals $\frac{2}{3}$. 
Iteration and Operations (Conceptual)

- Backward jumps are used to represent a subtraction operation.
- The expression $\frac{8}{8} - \frac{1}{8}$ represents the separating of one $\frac{1}{8}$ interval from the total.
- 7 intervals (or copies) of $\frac{1}{8}$ equals $\frac{7}{8}$. 

# Equivalent Fractions

Conceptual and Procedural

\[
\frac{2}{3}
\]

### Table

<table>
<thead>
<tr>
<th></th>
<th>Generate an equivalent fraction for the given fraction by multiplying by a fraction equal to 1.</th>
</tr>
</thead>
</table>
| 1. | \[
    \frac{2}{3} \times \frac{4}{4} = \frac{8}{12}
    \]

<table>
<thead>
<tr>
<th></th>
<th>Construct a number line to represent the given fraction, then partition to represent an equivalent fraction.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td><img src="image" alt="Double number line" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Write an equation to show the relationship between the given fraction and the equivalent fraction.</th>
</tr>
</thead>
</table>
| 3. | \[
    \frac{2}{3} = \frac{8}{12}
    \]                                                            |
Strand 3: Addition and Subtraction of Fractions (25 lessons)

- Composing and Decompose Fractions w/Addition and Subtraction
- Apply Properties of Operations with Fractions
- Generate Equivalent Fractions (conceptual and procedural)
- Add and Subtract Fractions with Like and Unlike Denominators
- Benchmark Comparisons (0, ½, 1)
- Solve Addition and Subtraction Fraction Story Problems
Composing and Decomposing Fractions with Addition

Understand how a fraction can be decomposed in multiple ways and represent the fraction as joining multiple parts of the unit fraction.

- The number line represents 2 whole units or the fraction $\frac{4}{2}$.

- Decompose the fraction $\frac{4}{2}$ in multiple ways using addition:
  
  \[
  \frac{4}{2} = \frac{2}{2} + \frac{2}{2} \\
  \frac{4}{2} = \frac{1}{2} + \frac{3}{2}
  \]
Composing and Decomposing Fractions with Subtraction

Understand how a fraction can be decomposed in multiple ways and represent the fraction as separating parts referring to the same whole.

- The number line represents 2 whole units or the fraction $\frac{8}{4}$.

- Write equations to show multiple ways to decompose the fraction $\frac{8}{4}$ using subtraction.

\[
\frac{8}{4} - \frac{2}{4} = \frac{6}{4}
\]

\[
\frac{8}{4} - \frac{7}{4} = \frac{1}{4}
\]

\[
\frac{8}{4} - \frac{5}{4} = \frac{3}{4}
\]
Understanding Mixed Numbers by Decomposing Improper Fractions

- Understand a fraction is equal to 1 when the numerator and denominator have the same value.
- Decompose an improper fraction into a fraction equal to a whole number plus a fraction less than 1

\[
\frac{7}{5} = \frac{5}{5} + \frac{2}{5}
\]

- Multiple ways to decompose fractions

- The unit fraction of \(\frac{7}{5}\) is \(\frac{1}{5}\).

- 7 intervals of \(\frac{1}{5}\) represents \(\frac{7}{5}\).

- 5 intervals of \(\frac{1}{5}\) represents \(\frac{5}{5} = 1\).
Benchmark Comparisons \( \frac{5}{8}, \frac{4}{6} \)

Decompose the fraction using the benchmark strategy.

\[
\begin{align*}
\frac{5}{8} & = \frac{4}{8} + \frac{1}{8} \\
\frac{4}{6} & = \frac{3}{6} + \frac{1}{6}
\end{align*}
\]
Using the Identity Property to Generate Common Denominators

Equation: \( 3 - \frac{1}{4} = N \)

1. Understand the identity property with whole numbers
   \[ 3 \times 1 = 3 \]

2. Identify the unit fraction
   \[ 3 \times \frac{4}{4} = 3 \]

3. Write a fraction equal to 1 using the unit fraction
   \[ \frac{3 \times 4}{1 \times 4} - \frac{1}{4} = N \]

4. Multiply the fraction equal to 1 and the whole number
   \[ \frac{12}{4} - \frac{1}{4} = \frac{11}{4} \]
Adding and Subtracting Fractions with Unlike Denominators

\[ \frac{3}{5} + \frac{1}{2} = \frac{6}{10} + \frac{5}{10} \]

عناصر من الاستشراق

\[ \frac{6}{10} + \frac{5}{10} \]

افتراض: الطلاب يمكن أن يستخدموا جدول الضرب لتحديد المعدول المشترك.

النقطة: الطلاب يمكن أن يستخدموا أي معدول على الرغم من أنهم يعرفون أن استخدام المعدول المشترك الأصغر يكون أسهل.

مواضيع المصدر: ضبط ما ينمو عن طريق التعلم من خلال استخدام جدول الضرب لتحديد المعدول المشترك.

النقطة: الطلاب يمكن أن يستخدموا أي معدول على الرغم من أنهم يعرفون أن استخدام المعدول المشترك الأصغر يكون أسهل.
Strand 4: Multiplication and Division of Fractions (30 lessons)

- Understand Fraction Multiplication and Division
  - Number line and area model
  - Relationship between multiplication and division
- Apply Properties of Operations with Fractions
- Multiply and Divide Fractions
- Predict the Size of Fraction Products/Quotients
- Solve Fraction Multiplication and Division Story Problems
Understanding Multiplication
Using the Number line

- Multiplication of a whole number by a fraction: \(2 \times \frac{3}{5} = N\)

- Repeated addition equation: \(2 \times \frac{3}{5} = \frac{3}{5} + \frac{3}{5} = \frac{6}{5}\)
Understanding Multiplication Using an Area Model

- Multiplication of a whole number by a fraction:
  \[ 3 \times \frac{5}{6} = N \]

- Repeated addition equation:
  \[ 3 \times \frac{5}{6} = \frac{5}{6} + \frac{5}{6} + \frac{5}{6} = \frac{15}{6} \] or \[ 2 \frac{3}{6} \]
## Zero Property
Any number multiplied by zero is zero

\[
5 \times 0 = 0 \quad \frac{3}{8} \times 0 = 0 \quad a \times 0 = 0
\]

## Identity Property
A number does not change when multiplied by 1

\[
a \times 1 = a
\]

<table>
<thead>
<tr>
<th>Whole Numbers</th>
<th>Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x 4 = 4</td>
<td>(\frac{3}{3} \times 4 = 4)</td>
</tr>
</tbody>
</table>

## Commutative Property
Changing order of factors does not change product

\[
a \times b = b \times a
\]

<table>
<thead>
<tr>
<th>Whole Numbers</th>
<th>Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 3 = 6</td>
<td>3 x 2 = 6</td>
</tr>
<tr>
<td>(\frac{1}{3} \times 2 = \frac{2}{3})</td>
<td>(2 \times \frac{1}{3} = \frac{2}{3})</td>
</tr>
</tbody>
</table>

## Associative Property
When 3 or more factors are multiplied the product is the same regardless of the grouping of the factors

\[
(a \times b) \times c = a \times (b \times c)
\]

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>(1 x 4) x 3 = 12</td>
<td>(1 x 3) x 4 = 12</td>
</tr>
</tbody>
</table>

## Distributive Property
The product of a number and a sum is equal to the sum of the individual products

\[
a \times (b + c) = (a \times b) + (a \times c)
\]

<table>
<thead>
<tr>
<th>Whole Numbers</th>
<th>Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x (3 + 5) = 16</td>
<td>(2 x 3) + (2 x 5) = 16</td>
</tr>
</tbody>
</table>

### Whole Numbers

<table>
<thead>
<tr>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

### Fractions

<table>
<thead>
<tr>
<th>(\frac{3}{3})</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\frac{1}{3})</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\frac{1}{3})</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\frac{1}{3})</th>
<th>(\frac{2}{3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{3})</td>
<td>(\frac{2}{3})</td>
</tr>
</tbody>
</table>
Relationship Between Multiplication and Division as a Strategy to Divide Fractions

\[ 5 \div \frac{1}{3} = N \quad N \times \frac{1}{3} = 5 \]

Inverse operation: Invert and Multiply

\[ 15 \times \frac{1}{3} = 5 \quad 5 \times \frac{3}{1} = 15 \]
PAR PILOT STUDY
PAR Year 3: Pilot

- Pilot Study October 2014 – June 2015
- Five school districts in Oregon
- Schools: 4 treatment, 3 control
- Classrooms: 6 treatment, 5 control
- 6th Grade Students: 110 treatment, 84 control
PAR Year 3: Pilot

• Research questions:
  – Did students in PAR treatment classrooms make greater gains on a battery of rational number assessments compared to peers in matched control classrooms?
  – Was PAR implemented as intended in treatment classrooms?
  – What were rates of specific teaching behaviors, hypothesized to positively impact student achievement, in treatment and control classrooms?
  – What were teachers’ and students’ perceptions of the PAR curriculum in terms of feasibility, usability, and student learning?
Pilot Demographic and Baseline Data

<table>
<thead>
<tr>
<th>Metric</th>
<th>Treatment</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>OAKS Assessment</td>
<td>218.92</td>
<td>218.36</td>
</tr>
<tr>
<td>Female</td>
<td>59.1</td>
<td>47.6</td>
</tr>
<tr>
<td>White</td>
<td>48.2</td>
<td>61.9</td>
</tr>
<tr>
<td>Hispanic or Latino</td>
<td>25.5</td>
<td>29.8</td>
</tr>
<tr>
<td>Black or African American</td>
<td>4.5</td>
<td>2.4</td>
</tr>
<tr>
<td>Other ethnicities</td>
<td>21.8</td>
<td>5.9</td>
</tr>
</tbody>
</table>

*Note.* No significant differences between treatment and control group scores on OAKS administered in the previous grade (5). A passing score on OAKS in grade 5, which indicates students are meeting standards, is 225. Students “nearly meet” standards with a score of 219.
Pilot Study Measurement Net

• **Student Assessments**
  – Proximal pre/post assessment at beginning and end of each curriculum strand implemented to assess mastery
  – Beginning and end of study:
    • EasyCBM, Grade 6
    • AIMSweb Math Computation, Grade 5
    • AIMSweb Math Computation, Grade 6
    • Algebra Readiness Progress Monitoring (ARPM: Ketterlin-Geller, 2013)
      – Quantity Discrimination
      – Number Properties
      – Proportional Reasoning
Pilot Study Measurement Net

• **Teacher surveys**
  – Demographic information and perceptions of the program
  – Mathematical Knowledge for Teaching at pre and posttest (Hall, Schilling & Ball, 2004)

• **Student group interviews**
  – Perceptions of the curriculum, its features, and their learning in PAR vs. other programs

• **Classroom observations**
  – Fidelity of implementation observations once per strand by curriculum team
  – Classroom Observations of Student Teacher Interactions – Cognitive Demand (COSTI-CD) observations by research staff to answer exploratory questions
## Pilot Study **Distal** Assessment Gain Scores

<table>
<thead>
<tr>
<th>Measure</th>
<th>Condition</th>
<th>N</th>
<th>M</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>EasyCBM, Grade 6</td>
<td>Treatment</td>
<td>93</td>
<td>2.66</td>
<td>4.84</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>71</td>
<td>-0.49</td>
<td>6.16</td>
</tr>
<tr>
<td>AIMSweb Grade 5</td>
<td>Treatment</td>
<td>94</td>
<td>-0.40</td>
<td>3.46</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>70</td>
<td>-0.31</td>
<td>3.60</td>
</tr>
<tr>
<td>AIMSweb Grade 6</td>
<td>Treatment</td>
<td>96</td>
<td>1.74</td>
<td>5.39</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>69</td>
<td>0.64</td>
<td>5.78</td>
</tr>
<tr>
<td>ARPM Quantity Discrimination</td>
<td>Treatment</td>
<td>103</td>
<td>0.98</td>
<td>7.46</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>72</td>
<td>1.26</td>
<td>6.97</td>
</tr>
<tr>
<td>ARPM Number Properties</td>
<td>Treatment</td>
<td>103</td>
<td>1.68</td>
<td>7.28</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>72</td>
<td>1.92</td>
<td>7.40</td>
</tr>
<tr>
<td>ARPM Proportional Reasoning</td>
<td>Treatment</td>
<td>103</td>
<td>1.16</td>
<td>7.17</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>72</td>
<td>0.14</td>
<td>6.51</td>
</tr>
</tbody>
</table>

*Note.* Gains reported reflect raw score changes from pretest to posttest.
### Pilot Study Proximal Assessment Raw Scores

<table>
<thead>
<tr>
<th>Measure</th>
<th>Time</th>
<th>Condition</th>
<th>N</th>
<th>M</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strand 1: Multiplication and Division of Whole Numbers</td>
<td>Pre</td>
<td>Treatment</td>
<td>102</td>
<td>4.25</td>
<td>2.48</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>74</td>
<td>3.32</td>
<td>1.96</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>Treatment</td>
<td>103</td>
<td>5.53</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>74</td>
<td>4.27</td>
<td>2.41</td>
<td></td>
</tr>
<tr>
<td>Strand 2: Fractions as Numbers</td>
<td>Pre</td>
<td>Treatment</td>
<td>102</td>
<td>4.50</td>
<td>1.55</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>73</td>
<td>4.23</td>
<td>1.80</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>Treatment</td>
<td>102</td>
<td>6.41</td>
<td>2.28</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>73</td>
<td>4.64</td>
<td>1.59</td>
<td></td>
</tr>
<tr>
<td>Strand 3: Addition and Subtraction of Fractions</td>
<td>Pre</td>
<td>Treatment</td>
<td>101</td>
<td>5.61</td>
<td>2.30</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>73</td>
<td>4.92</td>
<td>2.02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>Treatment</td>
<td>99</td>
<td>7.62</td>
<td>2.53</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>66</td>
<td>5.39</td>
<td>2.06</td>
<td></td>
</tr>
<tr>
<td>Strand 4: Multiplication and Division of Fractions</td>
<td>Pre</td>
<td>Treatment</td>
<td>74</td>
<td>5.08</td>
<td>2.03</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>67</td>
<td>3.82</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>Treatment</td>
<td>73</td>
<td>5.82</td>
<td>2.32</td>
</tr>
<tr>
<td></td>
<td>Control</td>
<td>71</td>
<td>4.10</td>
<td>1.99</td>
<td></td>
</tr>
</tbody>
</table>
## Tests of Statistical Significance

<table>
<thead>
<tr>
<th>Measure</th>
<th>$F$</th>
<th>$p$-value</th>
<th>Hedges’ $g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strand 1</td>
<td>0.70</td>
<td>0.27</td>
<td>0.25</td>
</tr>
<tr>
<td>Strand 2</td>
<td>1.68</td>
<td>0.03</td>
<td>0.84</td>
</tr>
<tr>
<td>Strand 3</td>
<td>1.59</td>
<td>0.02</td>
<td>0.68</td>
</tr>
<tr>
<td>Strand 4</td>
<td>0.43</td>
<td>0.24</td>
<td>0.20</td>
</tr>
<tr>
<td>AIMSweb Total*</td>
<td>0.08</td>
<td>0.46</td>
<td>0.15</td>
</tr>
<tr>
<td>ARPM Total</td>
<td>2.21</td>
<td>0.54</td>
<td>0.15</td>
</tr>
<tr>
<td>EasyCBM Total Score</td>
<td>3.23</td>
<td>0.11</td>
<td>0.44</td>
</tr>
</tbody>
</table>

*Note: AIMSweb analysis used ANCOVA, nesting students within schools. All other tests used ANOVA, nesting students within schools.

- Positive, significant effects on the Strand 2 and Strand 3 proximal assessments
- Substantively important effects (WWC, 2015) on the Strand 1 proximal assessment and EasyCBM
Other Results

• Teacher surveys
  – Teacher perception data being summarized for analysis
  – Little average change in teacher knowledge in the treatment or control group

• Student group interviews
  – In general, students liked features of the PAR curriculum: Scores trend toward 3 on a 4-point scale, where 4 is high
  – Students felt they learned from PAR: Scores trend toward 7.5 on a 10-point scale, where 10 is high

• Classroom observations
  – In process of summarizing for analysis
Pilot Study Implications

• The PAR curriculum demonstrated feasibility and social validity for use in middle school:
  – Trained teachers indicated PAR was feasible to implement in the classroom.
  – Students receiving intervention perceived the PAR curriculum as beneficial for learning math.

• The PAR curriculum demonstrated promise for improving math achievement among students in need of math support.
Pilot Study Caveats

• Students not randomly assigned to intervention

• Some differences in demographics between groups:
  – Treatment group was more female and more diverse ethnically

• Some treatment groups were unable to implement strand 4 of the curriculum

• Relatively small sample
Future Directions & Next Steps

- Disseminate results in publications
- Finalize curriculum for distribution via CTL Marketplace ([https://dibels.uoregon.edu/market/movingup/math](https://dibels.uoregon.edu/market/movingup/math))
- Propose an efficacy study of the program implemented on a larger scale using a more rigorous design
- Explore alternative use of PAR (e.g., by strand)
- Explore technology supplements to the intervention
Flexible Uses of PAR

• Pilot study examined use as a supplementary curriculum for students struggling in mathematics in sixth grade
• Standards addressed align with the development of rational number understanding from third grade through early middle school
• Potential for use as a supplemental program for students on track in elementary school, or for use with struggling students in older grades, by strand or in whole
Promoting Algebra Readiness: Developing a Strategic Intervention on Rational Number Concepts

Questions?
Nancy: nnelson3@uoregon.edu
Ben: clarkeb@uoregon.edu
Kathy: kjj@uoregon.edu

Thank You!
Other Materials

IES PRACTICE GUIDE ALIGNMENT

PAR aligns with these PG recommendations:
**Focus** intensely on in-depth treatment of rational numbers in grades 4-8; **Use** explicit and systematic instruction that includes models of proficient problem solving, verbalization of thought processes; guided practice, corrective feedback and frequent cumulative review; **Include** instruction on solving word problems based on common underlying structures; **Use** visual representations of mathematical ideas; **Monitor** student progress.
IES Practice Guide:  
https://ies.ed.gov/ncee/wwc/PracticeGuide/15

PAR aligns with these PG recommendations:  
Use equal sharing to introduce the concept of fractions dividing sets of objects as well as single whole objects; Help students recognize that fractions are numbers and they expand the number system beyond whole numbers; Use number lines as the central representational tool; Help student understand why procedures for computations with fractions make sense using area models, number lines and other visual representations; Provide opportunities for students to use estimation to predict/judge reasonableness; Address common misconceptions regarding procedures with fractions.

PAR aligns with these PG recommendations: Prepare problems and use them in whole-class instruction; Assist students in monitoring and reflecting on the problem-solving process; Teach students how to use visual representations; Expose students to multiple problem-solving strategies; Help students recognize and articulate mathematical concepts and notation.