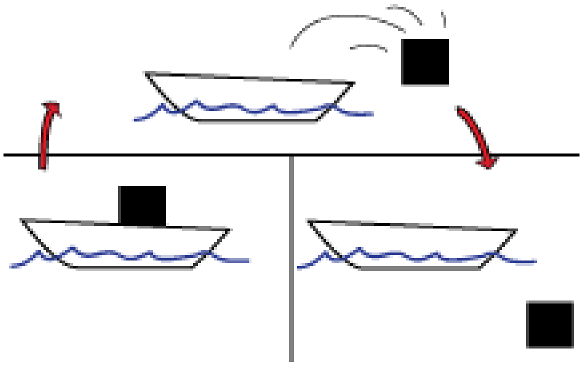
1 Problem



A boat is floating in water, carrying a box. The box is thrown overboard where it sinks. How does the volume of water displaced by the boat and box change?

2 Definitions

2.1 Variables

- ρ : density
- V : volume
- g : gravitational acceleration
- a : acceleration
- m : mass
- B : bouyant force
- W : weight

2.2 Markings

- x_w : water
- x_B : boat (V_B referrs to the volume of water displaced by the boat)
- x_b : box (V_b referrs to the volume of water displaced by the box)
- x_{Bb} : boat with the box in it (V_{Bb} refers to the volume of water displaced by the boat with the box in it)

2.3 Conventions

- The positive y direction is assumed to be up.

3 Calculations

The following set of equations represents the volume of water displaced by the boat with the box on it, V_{Bb}. It is assumed that the buoyant force is equal to the weight of the system.

$$\begin{aligned} B_{Bb} &= W_{Bb} \\ \rho_w V_{Bb} g &= (m_B + m_b) g \\ \rho_w V_{Bb} &= m_B + m_b \\ V_{Bb} &= \frac{m_B + m_b}{\rho_w} \end{aligned} \tag{1}$$

The following set of equations represents the volume of water displaced by the boat without the box, V_B. It is asumed that the buoyant force is equal to the weight of the boat.

$$\begin{aligned} B_B &= W_B \\ \rho_w V_B g &= m_B g \\ \rho_w V_B &= m_B \\ V_B &= \frac{m_B}{\rho_w} \end{aligned} \tag{2}$$

The following set of equations represents the volume of water displaced by the box in the water, V_b, in terms of buoyancy.

$$\begin{aligned} B_b - W_b &= -m_b a_b \\ \rho_w V_b g - m_b g &= -m_b a_b \\ \rho_w V_b - m_b &= m_b \left[\frac{-a_b}{g}\right] \\ \rho_w V_b &= m_b \left[\frac{-a_b}{g}\right] + m_b \\ \rho_w V_b &= m_b \left[1 - \frac{a_b}{g}\right] \\ V_b &= k\frac{m_b}{\rho_w} \ , \ k = 1 - \frac{a_b}{g} \end{aligned} \tag{3}$$

The numerical values of a_b and g are positive, so their ratio is a positive number. This means that $k < 1$.

The following set of equations represents the volume of water displaced by the box in the water, V_b, in terms of its mass and density.

$$\begin{aligned} \rho_b V_b &= m_b \\ V_b &= \frac{m_b}{\rho_b} \end{aligned} \quad (4)$$

The following set of equations represents the change in the volume of water displaced between the first state, V_1, when the box is in the boat, and the second state, V_2, when the box is overboard.

$$V_1 = V_{Bb} \quad , \quad V_2 = V_B + V_b$$
$$\begin{aligned} \Delta V &= V_2 - V_1 \\ \Delta V &= V_B + V_b - V_{Bb} \end{aligned} \quad (5)$$

The following set of equations solves for the change in volume of water displaced using the buoyancy representation of the volume of the box, V_b, from (3).

$$\begin{aligned} \Delta V &= V_B + V_b - V_{Bb} \\ \Delta V &= \frac{m_B}{\rho_w} + k\frac{m_b}{\rho_w} - \frac{m_B + m_b}{\rho_w} \\ \Delta V &= \frac{m_B + km_b - m_B - m_b}{\rho_w} \\ \Delta V &= \frac{m_b(k-1)}{\rho_w} < 0 \end{aligned} \quad (6)$$

Since $k < 1$, and $(k-1) < 0$, the change in volume is negative.

The following set of equations solves for the change in volume of the water displaced using the density representation of the volume of the box, V_b, from (4).

$$\begin{aligned} \Delta V &= V_B + V_b - V_{Bb} \\ \Delta V &= \frac{m_B}{\rho_w} + \frac{m_b}{\rho_b} - \frac{m_B + m_b}{\rho_w} \\ \Delta V &= \frac{m_B - m_B - m_b}{\rho_w} + \frac{m_b}{\rho_b} \\ \Delta V &= \frac{m_b}{\rho_b} - \frac{m_b}{\rho_w} < 0 \end{aligned} \quad (7)$$

Since the density of the box, ρ_b, must be greater than the density of the water, ρ_w, in order for the box to sink, the first fraction in (7) is smaller than the second, making the change in the volume negative.

4 Result

It has been found that the change in volume is negative, which means that the volume displaced by the boat with the box on it is higher than the volume displaced by the boat when the box is overboard. If the boat and box are located in a finite container, like a canal lock, the water level in the container will decrease when the box is thrown overboard.