Math and Decision Making (liberal arts math) Problem Sequence

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Notes are in red. Overall, this class was successful, although the sequence of problems can be improved. Two other University of Northern Iowa mathematics professors have taught the course using and modifying this sequence. The three of us will get together in April 2012 to create a refined list.

After the list is an appendix, intended for instructors, with notes on the voting topic.

Math 023: Comparing large sets.

I started with this topic because I wanted to kick off with something with which all the students were unfamiliar, so they felt that they were on “even territory.” And then, when we were finished with this topic, I informed them that this was considered very advanced mathematics, so they no longer could ever say, “I’m bad at math.”

1) Let S be the set of ways to arrange the letters ABC. Let T be the set of ways to pick an outfit, if I have three shirts (red, green, and blue) and two pairs of pants (blue, black). Which is larger, $|S|$ or $|T|$? Or are they the same?

2) Let S be the set of ways to arrange the letters ABCDEFGHIJ. Let T be the set of ways to arrange the letters KLMNOPQRST. Which is larger, $|S|$ or $|T|$? Or are they the same?

3) Let S be the set $\{1, 4, 7, 10, 13, 16, 19, ..., 3001\}$. Let T be the set $\{2, 5, 8, 11, 14, 17, 20, ..., 3002\}$. Which is larger, $|S|$ or $|T|$? Or are they the same?

4) Let S be the set of 10 letter words, where the first two letters are repeated, such as MMAQRESEDQ. Let T be the set of 9 letter words. Which is larger, $|S|$ or $|T|$? Or are they the same?

5) I have four boxes labeled A B C and D. A and B can hold four marbles, C can hold two, and D can hold 1. Let S be the number of ways to put 11 differently colored marbles into boxes. Let T be the number of ways to arrange the letters of MISSISSIPPI. Prove that $|S| = |T|$

6) Let S be all the ways to line up 8 /s and 3 *s in a row. These are different elements of S:

///*//*//*/
//*///*//*/
*/////**///

Let T be all the ways to write 8 as the sum of 4 positive numbers. These are different elements of T:

3 + 2 + 2 + 1 = 8
2 + 3 + 2 + 1 = 8
0 + 5 + 0 + 3 = 8

Prove that |S| = |T|

7) Let S equal the number of ways to line up *n* /s and *k* *s in a row. Let T equal the number of ways to write *n* as the sum of *k* + 1 positive numbers. Prove that |S| = |T|

8) Let (be the set of natural numbers: {1, 2, 3, 4, 5, 6, ...}. Let 2(be the set of even natural numbers {2, 4, 6, 8, 10, 12, ...} Prove that |(| = |2(|. It doesn't suffice to say "Infinity = infinity". Think of how you've done similar problems.

9) Let (be the set of natural numbers. Let W be the set of all words. Which is larger, |(| or |W|? Or are they the same?

10) Let (be the set of natural numbers. Let $($^{+}$ be the set of positive rational numbers, or positive numbers that can be written as fractions. (1/2 , 2/3, 22/7, 8, etc...) Which is larger, |(| or |$($^{+}$ |? Or are they the same?

Perhaps include: Show that the number of 8 digit numbers is the same as the number of 8 letter words using only the letters A – J and not beginning with A

11) In a city of 4000 people, we are choosing 3 to get a medal.

(a) Let S be the set of ways to pick 3 to get medals. Let T be the set of ways to pick 3997 *not* to get medals. Show that |S| = |T|.

(b) Let S be the set of ways to pick 3 to get medals. Let T be the set of ways to pick 3997 *to* get medals. Show that |S| = |T|.

12) We have a city of 4000 people. Let S be the set of ways to pick *n* of them to get medals. Let T be the set of ways to pick 4000 - *n* to get medals. Show that |S| = |T|.

13) One way to represent the number 10 is by a "dot picture." Here are some different "dot pictures" for the number 10:

OOO	OOO	OOO	OOOO	OOO	OO
OO	OOO	OO	OOO	OOO	OO
OOO	OOO	OO	O	OO	OO
OO		O	O	OO	O
	O	O	O		O
		O			O
					O

Notice that in a "dot picture", each row at least as many dots as the row below it.

Let S be the set of dot pictures *with three rows* that represent the number 10
.

Let T be the set of ways to represent 10 as the sum of three positive integers. Here are some different elements of T:

10 = 5 + 3 + 2 10 = 8 + 1 + 1
10 = 4 + 3 + 3 10 = 6 + 3 + 1
10 = 6 + 2 + 2 10 = 7 + 2 + 1

Notice that we are considering 10 = 5 + 3 + 2 to be "the same" element as 10 = 2 + 3 + 5. Also note that 10 = 9 + 1 + 0 is not part of T, because 0 is not a positive integer.

Show that |S| = |T|

14) Let S be the set of ways to represent 10 as the sum of four positive integers. Here are some different elements of S:

10 = 4 + 3 + 2 + 1 10 = 5 + 3 + 1 + 1
10 = 4 + 2 + 2 + 2 10 = 3 + 3 + 3 + 1
10 = 7 + 1 + 1 + 1 10 = 3 + 3 + 2 + 2

(as before, we are considering 10 = 4 + 3 + 2 + 1 to be "the same" element as 10 = 1 + 2 + 3 + 4)

Let T be the set of ways to represent 10 as the sum of natural numbers (not necessarily four of them) where the biggest number used is 4. Here are some different elements of T:

10 = 4 + 4 + 2
10 = 4 + 1 + 1 + 1 + 1 + 1 + 1
10 = 4 + 3 + 3
10 = 4 + 3 + 1 + 1 + 1

Show that |S| = |T|. (Hint: Dot pictures can be helpful here.)

15) Let W be set of words. Let T be the set of words beginning with 'A'. Show |W| = |T|. It does not suffice to say "infinity = infinity."

16) Let T = {3, 6, 9, 12, 15, 18, 21, …} Show ||=|T|.

17) Look at this table of fractions!

	1/1	2/1	3/1	4/1	5/1	6/1
	1/2	2/2	3/2	4/2	5/2	6/2
	1/3	2/3	3/3	4/3	5/3	6/3
	1/4	2/4	3/4	4/4	5/4	6/4
	1/5	2/5	3/5	4/5	5/5	6/5
	1/6	2/6	3/6	4/6	5/6	6/6

Prove that every element of appears at least once in the table.

18) Look at this list of numbers!

Use the result of the previous problem to show that every element of appears at least once in the list.

19) We thin the sequence above by only keeping the lowest terms fractions:

Prove that every element of appears exactly once in the new list.

20) Prove that |(| = |($^+$ |

21) Is the result from #20 amazing or what?

22) Prove that if S is countable, and T is countable, then |S| = |T|

23) The set is the set of all ordered pairs of natural numbers. Here are some members of : (1,2) (3,1) (9,2) etc.
Prove that is countable

24) Prove that the set of odd numbers is countable

25) Prove that the set of words is countable

26) Let ***I*** be the set of real numbers between zero and one. We are going to try to prove that ***I*** is countable. Don't worry, I'm not going to ask you to come up with a numbering scheme! But pretend we had one. Such a scheme might look like this:

Let A be the element of ***I*** defined as follows:

The first decimal place of A is the first decimal place of the first number in the list, incremented by one. The second decimal place of A is the second decimal place of the second number in the list, incremented by one. Etc. So in our example, A begins this way: 0.2146

(a) Write out the first six decimal places of A.

(b) Prove that A is not on the list

27) Use the previous idea to prove that ***I*** is NOT countable! Your proof might begin "Assume ***I*** was countable. Then we would be able to ..."

Possible problems to include in the future:

- Prove that if S is a subset of , then S is countable
- Prove that is countable

Math 023: pROBABILITY.

Counting wasn't a major part of this course, it is only used as a gateway to probability. In a course with an emphasis on counting, there would be more development in that area. (My colleagues and I have a sequence for a combinatorics course that we are polishing; contact me if you would like it to include)

28) I have three different T-shirts {red, yellow, blue}, two different pairs of pants {jeans, dark slacks}, and two different pairs of shoes {red sneakers, old flip-flops}. How many possible outfits do I have?

29) Gordon, my dad, my mom, and I compete in an essay contest. One of us will win "first place" and one of us will win "second place." How many possible ways are there to award the prizes?

30)

a) How many two letter words are there that only use the letters A, B, C and D?

b) How many three letter words are there that have the following three properties?

- ☠ The first letter is A, B or C
- ☠ The second letter is A or B
- ☠ The third letter is A or B

(Examples: CAB, AAB, BBA, CBA, etc...)

31) The answers to all the problems below are over 500.

a) How many two letter words are there (no restrictions on the letters)?

b) How many three letter words are there?

c) How many *k* letter words are there? (Where *k* is a variable standing for a natural number)

32) We roll a red die, a blue die, and a green die.

a) How many possible outcomes are there?

b) How many possible outcomes are there with all even numbers?

c) How many possible outcomes are there with exactly one 6?

33) A *vowel* is a letter from the set {A, E, I, O, U}. A *consonant* is a letter that is not a vowel.

a) How many 5 letter words are there that begin with a vowel?

b) How many 8 letter words are there that end with a consonant?

c) How many 10 letter words are there that either begin with a vowel or end with a vowel, but not both?

34) The set is the set of all ordered pairs consisting of an element of S followed by an element of T. For example, if and , then .

a) If $|S|= m$, and $|T| = n$, then what is

b) Justify your answer to part a)

35) One hundred students compete in an essay contest. There are prizes for “best essay”, “longest essay”, and “best grammar.” How many ways are there to award these prizes?

36) One hundred students compete in an essay contest. How many ways are there to award first, second, and third place prizes to the students?

37)

a) How many 8 letter words are there?

b) How many 8 letter words are there with no repeated letters?

38) Let k be some natural number.

a) How many k letter words are there?

b) How many k letter words are there with no repeated letters?

39) A flag is to be designed with 10 stripes. The stripes can be red, yellow, or blue. No stripe should be the same color as the one below it. How many ways are there to color the flag?

40) In the C++ programming language, a variable name must start with a letter or the underscore character (_) and succeeding characters must be letters, digits, or the underscore character. Uppercase and lowercase letters are considered to be different. How many variable names having at most four characters can be formed in C++?

41) How many 5 digit odd numbers are there that have distinct digits?

42) A music class of eight girls and seven boys is having a recital. If each member is to perform once, how many ways can the program be arranged in each of the following *cases?*

42a) The girls must all perform first

42b) A girl must perform first and a boy must perform last.

42c) Olivia will perform first, and Logan will perform last

42d) The entire program will alternate between girls and boys

42e) The first, eighth, and fifteenth performers must be girls

43) I have three shirts {T-shirt, dress shirt, puffy shirt} and 10 pairs of pants. Six pairs of pants go with my T-shirt, three pairs go with the dress shirt, and four pairs go with the puffy shirt. I also have three kinds of shoes, which go with everything.

43a) I said that I had 10 pairs of pants. But 6 + 3 + 4 is more than 10. Is there a problem? Why or why not?

43b) How many outfits do I have to choose from?

43c) This is the big one. In order to answer problem 43b (the answer is 39, by the way) you had to both add and multiply. Explain a rule that determines when you add, and when you multiply.

44) How many ways are there to choose k objects from a set of n objects, if order matters, and repetition is allowed?

45) How many ways are there to choose k objects from a set of n objects, if order matters and repetition is not allowed?

46) I have three different T-shirts {red, yellow, blue}, two different pairs of pants {blue jeans, black slacks}, and two different pairs of shoes {red sneakers, old green flip-flops}.

a) If I pick my clothes at random, what is the probability that I will be wearing something yellow?

b) If I pick my clothes at random, what is the probability that I will be wearing something red?

47) Gordon, my dad, my mom, and I compete in an essay contest. One of us will win "first place" and one of us will win "second place."

a) If the prizes are awarded at random, what is the probability that I will win first place and Gordon will win second?

b) What is the probability my dad will win first place?

c) What is the probability my mom will win a prize?