Number Theory A

1. Find the last three digits of

$$2008^{2007^{\cdot^{\cdot^{\cdot^{2^{1}}}}}}.$$

2. Find the largest integer n which equals the product of its leading digit and the sum of its digits.

3. In how many ways can $1 + 2 + \cdots + 2007$ be expressed as a sum of consecutive positive integers?

4. A positive integer is called *squarefree* if its only perfect square factor is 1. Call a set of positive integers *squarefreeful* if each product of two of its elements is squarefree, and *squarefreefullest* if no positive integer less than the maximum element of the set can be added while preserving the set's squarefreefulness. What is the minimum number of elements in a squarefreefullest set containing 31?

5. Let F_n be the Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. For each i, $1 \leq i \leq 200$, we calculate the greatest common divisor g_i of f_i and f_{2007}. What is the sum of the distinct values of g_i?

6. Find the number of ordered triplets of nonnegative integers (m, n, p) such that $m + 3n + 5p \leq 600$.

7. How many ordered pairs of integers (x, y) satisfy

$$8(x^3 + x^2y + xy^2 + y^3) = 15(x^2 + y^2 + xy + 1)?$$

8. For how many ordered pairs of positive integers (x, y) is $\frac{x^2+y^2}{x-y}$ an integer factor of 2310?

9. How many pairs of integers a and b are there such that a and b are between 1 and 42 and $a^9 = b^7$ mod 43?

10. Find all primes p such that there exists positive integers q and r such that $p \nmid q$, $3 \nmid q$, $p^3 = r^3 - q^2$