PUMaC 2008-9

Geometry

1. (2 points) What is the area of a circle with a circumference of 8?

2. (2 points) Consider a convex polygon $\mathcal{P}$ in space with perimeter 20 and area 30. What is the volume of the locus of points that are at most 1 unit away from some point in the interior of $\mathcal{P}$?

3. (3 points) Consider a 12-sided regular polygon. If the vertices going clockwise are A, B, C, D, E, F, etc, draw a line between A and F, B and G, C and H, etc. This will form a smaller 12-sided regular polygon in the center of the larger one. What is the area of the smaller one divided by the are of the larger one?

4. (3 points) How many ordered pairs of real numbers (x, y) are there such that $x^2 + y^2 = 200$ and
$$\sqrt{(x-5)^2 + (y-5)^2} + \sqrt{(x+5)^2 + (y+5)^2}$$
is an integer?

5. (4 points) Infinitesimal Randall Munroe is glued to the center of a pentagon with side length 1. At each corner of the pentagon is a confused infinitesimal velociraptor. At any time, each raptor is running at one unit per second directly towards the next raptor in the pentagon (in counterclockwise order). How far does each confused raptor travel before it reaches Randall Munroe?

6. (4 points) Find the coordinates of the point in the plane at which the sum of the distances from it to the three points $(0, 0)$, $(2, 0)$, $(0, \sqrt{3})$ is minimal.

7. (5 points) Let $\mathcal{H}$ be the region of points (x, y), such that $(1, 0)$, (x, y), $(-x, y)$, and $(-1, 0)$ form an isosceles trapezoid whose legs are shorter than the base between (x, y) and $(-x, y)$. Find the least possible positive slope that a line could have without intersecting $\mathcal{H}$.

8. (5 points) In four-dimensional space, the 24-cell of sidelength $\sqrt{2}$ is the convex hull of (smallest convex set containing) the 24 points $(\pm 1, \pm 1, 0, 0)$ and its permutations. Find the four-dimensional volume of this region.

9. (7 points) In tetrahedron $ABCD$ with circumradius 2, $AB = 2$, $CD = \sqrt{7}$, and $\angle ABC = \angle BAD = \frac{\pi}{2}$. Find all possible angles between the planes containing ABC and ABD.

10. (7 points) A cuboctahedron is the convex hull of (smallest convex set containing) the 12 points $(\pm 1, \pm 1, 0)$, $(\pm 1, 0, \pm 1)$, $(0, \pm 1, \pm 1)$. Find the cosine of the solid angle of one of the triangular faces, as viewed from the origin. (Take a figure and consider the set of points on the unit sphere centered on the origin such that the ray from the origin through the point intersects the figure. The area of that set is the solid angle of the figure as viewed from the origin.)