PUMaC 2008-9

Team A

1. (2 points) What is the difference between the median and the mean of the following data set: 12, 41, 44, 48, 47, 53, 60, 62, 56, 32, 23, 25, 31?

2. (2 points) Quadrilateral $ABCD$ has both an inscribed and a circumscribed circle and sidelengths $\overline{BC} = 4$, $\overline{CD} = 5$, $\overline{DA} = 6$. Find the area of $ABCD$.

3. (3 points) The seven dwarves are at work on day when they find a large pile of diamonds. They want to split the diamonds evenly among them, but find that they would need to take away one diamond to split into seven equal piles. They are still arguing about this when they get home, so Snow White sends them to bed without supper. In the middle of the night, Sneezy wakes up and decides that he should get the extra diamond. So he puts one diamond aside, splits the remaining ones in to seven equal piles, and takes his pile along with the extra diamond. Then, he runs off with the diamonds. His sneeze wakes up Grumpy, who, thinking along the same lines, removes one diamond, divides the remainder into seven equal piles, and runs off. Finally, Sleepy, for the first time in his life, wakes up before sunrise and performs the same operation. When the remaining four dwarves arise, they find that the remaining diamonds can be split into 5 equal piles. Doc suggests that Snow White should get a share, so they have no problem splitting the remaining diamonds. Happy, Dopey, Bashful, Doc, and Snow White live happily ever after. What's the smallest possible number of diamonds that the dwarves could have started out with?

4. (3 points) The graphs of the following equations divide the xy plane into some number of regions.

$$4 + (x+2)y = x^2$$
$$(x+2)^2 + y^2 = 16$$

Find the area of the second smallest region.

5. (4 points) Alice and Bob play a game with a coin. The coin is thrown and Alice wins \$1 from Bob if it lands heads, and Bob \$1 from Alice if it lands tails. Initially, Alice has 2008 dollars and Bob 2009. Find the sum of the probability that Alice wins and the expected number of tosses until one of them wins, expressed as a mixed number.

6. (4 points) Suppose that the roots of the quadratic $x^2 + ax + b$ are α and β. Then α^3 and β^3 are the roots of some quadratic $x^2 + cx + d$. Find c in terms of a and b.

7. (5 points) Alex Lishkov is trying to guess sequence of 2009 random ternary digits (0, 1, or 2). After he guesses each digit, he finds out whether he was right or not. If he guesses incorrectly, and k was the correct answer, then an oracle tells him what the next k digits will be. Being Bulgarian, Lishkov

plays to maximize the expected number of digits guessed correctly. Let P_n be the probability that Lishkov guesses the nth digit correctly. Find P_{2009}. Write your answer in the form $x + y\text{Re}(\rho^k)$, where x and y are rational, ρ is complex, and k is a positive integer.

8. (5 points) Let a and b be the two complex roots of the polynomial $2x^3 + 2x + 1$. Find the smallest degree monic polynomial with integer coefficients with a/b as a root.

9. (7 points) Let $X = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{100}$. If p is the polynomial of minimal degree with integer coefficients such that $p(X) = 0$, find the prime factorization of the sum of the roots of p.

10. (7 points) Consider the sequence $s_0 = (1, 2008)$. Define new sequences s_i inductively by inserting the sum of each pair of adjacent terms in s_{i-1} between them—for instance, $s_1 = (1, 2009, 2008)$. For some n, s_n has exactly one term that appears twice. Find this repeated term.