PUMaC 2008-9

Team

1. Calculate $\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}+\frac{6}{1+\frac{6}{1+\ldots}}$.

(**ANS:** 5 Let $X=\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$ and $Y=\frac{6}{1+\frac{6}{1+\ldots}}$. We wish to find $X+Y$. Now we have $X^2=6+X$, whence $X=(1+\sqrt{1+24})/2=3$. Also, $Y=\frac{6}{1+Y}$, so $Y(1+Y)=6$, thus $Y^2+Y-6=0$ whence $Y=(-1+\sqrt{1+24})/2=2$. Thus we conclude that $X+Y=3+2=5$ as claimed. **CB:** GBH?, ACH)

2. Find $\log_2 3 * \log_3 4 * \log_4 5 * \ldots * \log_{62} 63 * \log_{63} 64$.

(**ANS:** 6 **CB:** GBH?, ACH)

3. What is the smallest positive integer value of x for which $x \equiv 4 \pmod 9$ and $x \equiv 7 \pmod 8$?

(**ANS:** 31 **CB:** GBH?, ACH)

4. What is the difference between the median and the mean of the following data set: 12, 41, 44, 48, 47, 53, 60, 62, 56, 32, 23, 25, 31?

(**ANS:** $2.923076 = 38/13$ **CB:** TW, PR, ACH)

5. Quadrilateral $ABCD$ has both an inscribed and a circumscribed circle and sidelengths $\overline{BC}=4$, $\overline{CD}=5$, $\overline{DA}=6$. Find the area of $ABCD$.

(**ANS:** $10\sqrt{6}$
Because the quadrilateral has an inscribed circle, pair of opposite sides must add up to the same thing, so $\overline{AB}=5$. Half the perimeter is $s=10$. This also has a circumscribed circle, so using Brahmagupta's formula for the area of a quadrilateral inscribed in a circle, the area is $\sqrt{(s-\overline{AB})(s-\overline{BC})(s-\overline{CD})(s-\overline{DA})}=\sqrt{5\cdot 6\cdot 5\cdot 4}=10\sqrt{6}$. Alternately, the quadrilateral is an isosceles trapezoid, whose height is easily calculable as $\sqrt{24}$ by the Pythagorean Theorem and the average of whose bases is 5.
CB: AL6, IAF, ACH)

6. The seven dwarves are at work on day when they find a large pile of diamonds. They want to split the diamonds evenly among them, but find that they would need to take away one diamond to split into seven equal piles. They are still arguing about this when they get home, so Snow White sends them to bed without supper. In the middle of the night, Sneezy wakes up and decides that he

should get the extra diamond. So he puts one diamond aside, splits the remaining ones in to seven equal piles, and takes his pile along with the extra diamond. Then, he runs off with the diamonds. His sneeze wakes up Grumpy, who, thinking along the same lines, removes one diamond, divides the remainder into seven equal piles, and runs off. Finally, Sleepy, for the first time in his life, wakes up before sunrise and performs the same operation. When the remaining four dwarves arise, they find that the remaining diamonds can be split into 5 equal piles. Doc suggests that Snow White should get a share, so they have no problem splitting the remaining diamonds. Happy, Dopey, Bashful, Doc, and Snow White live happily ever after. What's the smallest possible number of diamonds that the dwarves could have started out with?

(**ANS:** 337
We have $N = 7k + 1$.
After Sneezy, $6k$ remain.
We see $6k \equiv 1 \mod 7$, so $k = 7q + 6$. Thus there are $42q + 35 + 1$ diamonds. Then remaining are $36q + 30$ diamonds after Grumpy takes his.
We see $36q + 30 \equiv 1 \mod 7$, so $q \equiv -1 \mod 7$, so $q = 7a + 6$. Thus there are $252a + 216 + 30$ diamonds, and Sleepy takes this minus one divided by seven, leaving $6(252a + 245)/7$ diamonds remaining. This is $(1512a + 1470)/7 = 216a + 210$ diamonds remaining. This is divisible by 5. Thus we see 5 divides a. Thus $a = 0$ is the lowest possible. In summary, we have $a = 0$, $q = 6$, $k = 48$, so $N = 337$. **CB:** TL, JVP)

7. The graphs of the following equations divide the xy plane into some number of regions.

$$4 + (x + 2)y = x^2$$
$$(x + 2)^2 + y^2 = 16$$

Find the area of the second smallest region.

(**ANS:** $4\pi + 8$
Rewriting, we find: $(x + 2)y = (x - 2)(x + 2)$ to get $x = -2$ or $y = x - 2$. The second equation is a circle centered at $(-2, 0)$ with radius 4. The second smallest region is given by (x, y) with $x > -2$, $y > x - 2$, and $(x + 2)^2 + y^2 < 16$. This region can be divided into a quarter circle and an isoceles right triangle. Thus the area is $(1/4)\pi(4)2 + \frac{1}{2}(4)2 = 4\pi + 8$. **CB:** JVP)

8. Alice and Bob play a game with a coin. The coin is thrown and Alice wins \$1 from Bob if it lands heads, and Bob \$1 from Alice if it lands tails. Initially, Alice has 2008 dollars and Bob 2009. Find the sum of the probability that Alice wins and the expected number of tosses until one of them wins, expressed as a mixed number.

(**ANS:** $\frac{2008}{4017} + 4034072$. Let $f(n)$ denote the probability that Alice wins if she has n dollars, and let $g(n)$ denote the expected time until the game ends. Then $f(0) = 0$, $f(n) = \frac{f(n-1)+f(n+1)}{2}$, and $f(4017) = 1$. Unwinding the algebraic identity, we have $f(n+2) - f(n+1) = f(n+1) - f(n)$, so differences are constant and thus you have a linear function, and so the answer for the first part is $\frac{2008}{4017}$. g satisfies $g(0) = g(4017) = 0$ and $g(n) = \frac{g(n+1)+g(n-1)}{2} + 1$. Thus, $g(n+1) - g(n) = g(n) - g(n-1) - 2$, and so, by successive differences, g is a quadratic with leading term $-n2$ and roots at 0 and 4017, so $g(n) = n(4017 - n)$. **CB:** AL)

9. Suppose that the roots of the quadratic $x^2 + ax + b$ are α and β. Then α^3 and β^3 are the roots of some quadratic $x^2 + cx + d$. Find c in terms of a and b.

(**ANS:** $a^3 - 3ab$.
We know that $a = -\alpha - \beta$ and $b = \alpha\beta$. We wish to find $c = -\alpha^3 - \beta^3$. Now observe $c = -\alpha^3 - \beta^3 = -(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2) = -(\alpha+\beta)((\alpha+\beta)^2 - 3\alpha\beta) = -(-a)((-a)^2 - 3b) = a^3 - 3ab$.
CB: NS, JP, ACH, IAF)

10. Alex Lishkov is trying to guess sequence of 2009 random ternary digits (0, 1, or 2). After he guesses each digit, he finds out whether he was right or not. If he guesses incorrectly, and k was the correct answer, then an oracle tells him what the next k digits will be. Being Bulgarian, Lishkov plays to maximize the expected number of digits guessed correctly. Let P_n be the probability that Lishkov guesses the nth digit correctly. Find P_{2009}. Write your answer in the form $x + y\text{Re}(\rho^k)$, where x and y are rational, ρ is complex, and k is a positive integer.

(**ANS:**
We can prove by induction on the number of digits remaining that it is best for Alex to guess 0 when he doesn't know the digit and give the correct answer when he does. The base cases where he has 3 or fewer digits left can easily be checked by hand. For the other cases, If he doesn't know the answer, he guesses 0 because it maximizes the number of digits he will find out if he is wrong (except at the very last digit, where it doesn't matter what he guesses). If he does know the answer, by guessing correctly and following the given strategy, he can guarantee himself an expected value of at least $\frac{19}{9}$ and possibly some knowledge over the next three digits. By guessing incorrecly, he finds out at most two digits, which, by the inductive hypothesis, guarantees him an expected value of 2 over those three digits, and no knowledge at the end. So it is strictly better to guess correctly.
With this strategy, let a_n, b_n, and c_n be the probability that he gets the nth digit right given that it is a 0, 1, or 2, respectively. He always gets zeroes right, so $a_n = 1$. He gets a 1 or a 2 only when he knows it ahead of time, so it doesn't matter whether it's a 1 or a 2: $b_n = c_n$. The only time he can guess a 1 correctly is when he knows it, for which there are three mutually exclusive possibilities: He guessed incorrectly on a 1 or a 2 one step ago, or he guessed incorrectly on a 2 two steps ago. $b_n = \frac{2}{3}(1 - b_{n-1}) + \frac{1}{3}(1 - b_{n-2}) = 1 - \frac{2}{3}b_{n-1} - \frac{1}{3}b_{n-2}$. (Here we take $b_k = 1$ for

$k \leq 0$) So taking $B_n = b_n - 1/2$, we get $B_n = -\frac{2}{3}B_{n-1} - \frac{1}{3}B_{n-2}$. So, if r_+ and r_- are the roots of $x^2 = -\frac{2}{3}x - \frac{1}{3}$, then $B_n = A_+r_+^n + A_-r_-^n$. $r_\pm = \frac{-1\pm\sqrt{-2}}{3}$, and $B_0 = \frac{1}{2}$, $B_1 = -\frac{1}{2}$. So $\frac{1}{2} = A_+ + A_-$ and $-\frac{1}{2} = A_+\frac{-1+\sqrt{-2}}{3} + A_-\frac{-1-\sqrt{-2}}{3}$. So $A_\pm = \frac{1\pm\sqrt{-2}}{4}$. Note that $A_- = \overline{A_+}$ and $r_- = \overline{r_+}$, so $A_-r_-^n = \overline{A_-r_-^n}$. So, $B_n = A_+r_+^n + A_-r_-^n = 2\mathrm{Re}\left(A_+r_+^n\right)$. So, $B_{2009} = 2\mathrm{Re}\left(\frac{1+\sqrt{-2}}{4}\left(\frac{-1+\sqrt{-2}}{3}\right)^{2009}\right) = -\frac{1}{2}\mathrm{Re}\left(\left(\frac{-1+\sqrt{-2}}{3}\right)^{2008}\right)$. So the probability of guessing correctly on the 2009th digit is $\frac{1}{3}(1+2b_{2009}) = \frac{2}{3}(1+B_{2009}) = \frac{2}{3} - \frac{1}{3}\mathrm{Re}\left(\left(\frac{-1+\sqrt{-2}}{3}\right)^{2008}\right)$ **CB:** AM)

11. Let a and b be the two complex roots of the polynomial $2x^3+2x+1$. Find the smallest degree monic polynomial with integer coefficients with a/b as a root.

(**ANS:** $x^6+3x^5+10x^4+15x^3+10x^2+3x+1$. The basic idea is that, if you let the roots be a, b, and c, then you expand out $\left(x-\frac{a}{b}\right)\left(x-\frac{a}{c}\right)\left(x-\frac{b}{a}\right)\left(x-\frac{b}{c}\right)\left(x-\frac{c}{a}\right)\left(x-\frac{c}{b}\right)$. You get a holy mess that can be reduced to a polynomial in symmetric polynomials and powers of x. **CB:** IAF)

12. Let $X = \sqrt{1}+\sqrt{2}+\cdots+\sqrt{100}$. If p is the polynomial of minimal degree with integer coefficients such that $p(X)=0$, find the prime factorization of the sum of the roots of p.

(**ANS:** $2^{25}\cdot 5\cdot 11$

There are 25 prime numbers less than 100, $p_1,\ldots,p_n$. We may write $X = 1+2+3+4+5+6+7+8+9+10+\sum_{2\leq n\leq 99 \text{ not a square}}\sqrt{n} = 55+\sum_{2\leq n\leq 99 \text{ not a square}}\sqrt{n}$. Given any 25-tuple $\epsilon_i \in \{-1,1\}^{25}$, let $X_{\boldsymbol{\epsilon}}$ be X where we replace every occurence of $\sqrt{p_i}$ with $\epsilon_i\sqrt{p_i}$ for all i. Now suppose $P(T)$ is the minimal polynomial for $X = X_{1^{25}}$. Then when we multiply out $P(X)$, all the $\sqrt{p_i}$ must cancel. Thus if we change the sign of each occurence of $\sqrt{p_i}$, we do not change $P(X)$. Thus we conclude that for all $\boldsymbol{\epsilon}$, we have $P(X_{\boldsymbol{\epsilon}})=0$. Thus P has at least 2^{25} roots. Conversely, let us show that $P(T)=\prod_{\boldsymbol{\epsilon}}(T-X_{\boldsymbol{\epsilon}})$ has rational coefficients, and thus is the minimal polynomial of X. To see this, observe that the coefficients of $P(T)$ are polynomials in 1 and $\sqrt{p_i}$. However, we see that changing the sign on every occurence of $\sqrt{p_i}$ just permutes the roots of $P(T)$, and thus does not change its coefficients. Thus we conclude that the coefficients of $P(T)$ actually do not involve $\sqrt{p_i}$. Hence we see that $P(T)$ has rational coefficients as claimed. Now we just need to calculate the sum of the roots of P. But this is now easy: we get $2^{25}\cdot 55$, since the stuff in the sum $\sum_{2\leq n\leq 99 \text{ not a square}}\sqrt{n}$ all cancels when summed over all sign flips $\boldsymbol{\epsilon}$. Thus the answer is $2^{25}\cdot 5\cdot 11$ as claimed. **CB:** IAF, JVP)

13. Consider the sequence $s_0 = (1, 2008)$. Define new sequences s_i inductively by inserting the sum of each pair of adjacent terms in s_{i-1} between them—for instance, $s_1 = (1, 2009, 2008)$. For some n, s_n has exactly one term that appears twice. Find this repeated term.

(**ANS:** 1805535.

Instead of starting with 1 and 2008, use ordered pairs (1,0) and (0,1). (In s_2, for instance, you have (1,0),(2,1),(1,1),(1,2),(0,1).) By induction, after n iterations the greatest number that appears in any ordered pair is F_{n+1}. Two pairs (a, b) and (c, d) correspond to the same integers iff $2008(a - c) + 1(b - d) = 0$, so to have equal pairs we need some element at least 2008 (we never get the same pair twice because, if we view each ordered pair (a, b) as a fraction a/b, they're strictly decreasing). Since $F_{17} = 1597 < 2008$, we need at least 17 iterations.

Note that by induction, if n is the sum of the coefficients of the continued fraction expansion of a/b, then (a, b) first appears in s_n. Hence (2351, 898) first appears in s_{17}, and $(343, 899)$ first appears in s_{16}, since their continued fraction expansions are (2;1,1,1,1,1,1,1,1,1,1,1,2,1,1) and (0;2,1,1,1,1,1,3,3,1,1,1), respectively. These both correspond to 1805535, so that's the answer.

The continued fraction expansions also give a clue how to find these (the following section will be most useful for readers who've already played around with the sequences): if (a, b) and (c, d) are the above integers, we saw above that we needed a close to the maximum possible, which occurs only if $a \approx \phi b$ or $a \approx \phi^2 b$. Hence we must have either $(a, b, c, d) \approx (\phi b, b, \phi b - 2008, b + 1)$ or $(a, b, c, d) \approx (\phi^2 b, b, \phi^2 b - 2008, b + 1)$. Experimentation shows that we also want $\frac{d}{c}$ to be approximately ϕ or ϕ^2; of these cases the minimum value of b occurs when $\frac{a}{b} \approx \frac{d}{c} \approx \phi^2$. In this case, $\frac{b+1}{\phi^2 b - 2008} \approx \phi^2$ gives $b \approx \frac{2008\phi^2 - 1}{\phi^4 - 1} \approx 898$. In that case, $a \approx 2351$, $c \approx 343$, and $d \approx 899$, and in fact these values work.

All of the above was done by hand; proving that there is only one repeated value in s_{2007} required a computer search.

Thanks to Mathew Crawford for his contribution. **CB:** ACH)