PUMaC 2008-9

Individual Finals

1. Find all pairs of positive real numbers (a, b) such that $\frac{n-2}{a} \leq \lfloor bn \rfloor < \frac{n-1}{a}$ for all positive integers n.

(**ANS:** None exist. If $b > 0$, then we have $0 \leq \lfloor b \rfloor < \frac{1-1}{a} = 0$, a contradiction. **CB:** EK)

2. Let $\mathcal{P}$ be a convex polygon, and let $n \geq 3$ be a positive integer. On each side of $\mathcal{P}$, erect a regular n-gon that shares that side of $\mathcal{P}$, and is outside $\mathcal{P}$. If none of the interiors of these regular n-gons overlap, we call $\mathcal{P}$ *n-good*.

   (a) Find the largest value of n such that every convex polygon is n-good.

   (b) Find the smallest value of n such that no convex polygon is n-good.

(**ANS:** The angle measure of a regular n-gon is $(1 - \frac{2}{n})180°$. A polygon is n-good if and only if we can fit two of these angles outside each angle of the polygon; that is, if and only if $2(1 - \frac{2}{n})180° + x \leq 360°$ for each angle x of the polygon. Simplifying the inequality, we have $x \leq \frac{720°}{n}$. (a) The answer is 4. For $n = 4$, we have $x \leq 180°$, which is certaintly true for each angle of a convex polygon. For $n = 5$, we have $x \leq 144°$, so no polygon with an angle larger than 144° is 5-good. (b) The answer is 13. An equilateral triangle is n-good for all $3 \leq n \leq 12$. We now prove no polygon is 13-good. Suppose a polygon has k sides. The sum of the angles of the polygon is $(k - 2)180°$, so there must be some angle which measures at least $\frac{(k-2)180°}{k} = (1 - \frac{2}{k})180°$. Since $k \geq 3$, there must be angle measuring at least $(1 - \frac{2}{k})180° \geq (1 - \frac{2}{3})180° = 60°$. But the condition for 13-goodness is $x \leq \frac{720°}{13} < 60°$ for all angles x, so no polygon is 13-good. **CB:** GL)

3. A *hypergraph* consists of a set of vertices V and a set of subsets of those vertices, each of which is called an edge. (Intuitively, it's a graph in which each edge can contain multiple vertices). Suppose that in some hypergraph, no two edges have exactly one vertex in common. Prove that one can color this hypergraph's vertices such that every edge contains both colors of vertices.

(**ANS:** Suppose not. Consider a coloring of the vertices such that as few edges as possible are monochromatic, and consider one of its monochromatic edges e. Change the color of one vertex v of e. Any other vertex that contains v must also contain another vertex of e, which remains its original color, so this does not create any new monochromatic edges, but it makes e multicolored, so the coloring did not have a minimal number of monochromatic edges, contradiction. Hence the desired coloring is possible. **CB:** ACH)

4. Find all positive real numbers b for which there exists a positive real number k such that $n - k \leq \lfloor bn \rfloor \leq n$ for all positive integers n.

(**ANS:** The answer is 1. Let $b = q + r$, where q is an integer and $0 \leq r < 1$. Then $\lfloor bn \rfloor = qn + \lfloor rn \rfloor$. Suppose $q = 0$. Letting $r = 1 - s$, where $0 < s \leq 1$, we have $rn = n - sn$. For sufficiently large n, we can make sn larger than any k, so there are no solutions for $q = 0$. Now suppose $q \geq 1$. The inequality $qn + \lfloor rn \rfloor \leq n$ becomes $(q-1)n + \lfloor rn \rfloor \leq 0$. Both terms on the left hand side are nonnegative, so this can only be true if both are 0 for all n; that is $q = 1$ and $r = 0$. **CB:** GL)

5. A *hypergraph* consists of a set of vertices V and a set of subsets of those vertices, each of which is called an edge. (Intuitively, it's a graph in which each edge can contain multiple vertices). Suppose that in some hypergraph, no two edges have exactly one vertex in common. Prove that one can color this hypergraph's vertices such that every edge contains both colors of vertices.

   (**ANS:** Suppose not. Consider a coloring of the vertices such that as few edges as possible are monochromatic, and consider one of its monochromatic edges e. Change the color of one vertex v of e. Any other vertex that contains v must also contain another vertex of e, which remains its original color, so this does not create any new monochromatic edges, but it makes e multicolored, so the coloring did not have a minimal number of monochromatic edges, contradiction. Hence the desired coloring is possible. **CB:** ACH)

6. A sequence $\{a_i\}$ is defined by $a_1 = c$ for some $c > 0$ and $a_{n+1} = a_n + \frac{n}{a_n}$. Prove that $\frac{a_n}{n}$ converges and find its limit.

   (**ANS:** Let $b_n = a_n, n > 0$. Then one can check that $b_{n+1} = \frac{1}{n+1}(nb_n + \frac{1}{b_n})$. We know that $b_1 = a_1 = c + \frac{1}{c} \geq 2 > 1$. Then we have, if $b_n > 1$, $b_n > \frac{1}{b_n}$, so $b_{n+1} < \frac{1}{n+1}(nb_n + b_n) = b_n$ and $b_{n+1} > \frac{1}{n+1}((n-1) + b_n + \frac{1}{b_n} \geq 1$, so we then have $1 < b_{n+1} < b_n$, so the sequence converges. Now, to show that $\lim_{n\to\infty} b_n = 1$, we let $c_n = b_n - 1$. One can check that $c_{n+1} = \frac{1}{n+1}(nc_n - \frac{c_n}{c_n+1}$. Since $b_n > 1$, $c_n < \frac{n-1}{n}c_{n-1}$, and thus, by induction, $c_n < \frac{c_1}{n}$. Thus, $c_n \to 0$ and thus $b_n \to 1$, as claimed. **CB:** AL12, EPK)