PUMaC 2009

Algebra A

1. Find the root that the following three polynomials have in common:

$$x^3 + 41x^2 - 49x - 2009$$

$$x^3 + 5x^2 - 49x - 245$$

$$x^3 + 39x^2 - 117x - 1435$$

2. Given that $P(x)$ is the least degree polynomial with rational coefficients such that

$$P(\sqrt{2} + \sqrt{3}) = \sqrt{2},$$

find $P(10)$.

3. Let $x_1, x_2, ...x_{10}$ be non-negative real numbers such that $\frac{x_1}{1} + \frac{x_2}{2} + ... + \frac{x_{10}}{10} \leq 9$. Find the maximum possible value of $\frac{{x_1}^2}{1} + \frac{{x_2}^2}{2} + ... + \frac{{x_{10}}^2}{10}$.

4. Find the smallest positive α (in degrees) for which all the numbers

$$\cos\alpha, \cos 2\alpha, \ldots, \cos 2^n\alpha, \ldots$$

are negative.

5. Find the maximal positive integer n, so that for any real number x we have $\sin^n x + \cos^n x \geq \frac{1}{n}$.

6. Find the number of functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ for which $f(h+k) + f(hk) = f(h)f(k) + 1$, for all integers h and k.

7. Let $x_1, x_2, \ldots, x_n$ be a sequence of integers, such that $-1 \leq x_i \leq 2$, for $i = 1, 2, \ldots, n$, $x_1 + x_2 + \cdots + x_n = 7$ and $x_1^8 + x_2^8 + \ldots x_n^8 = 2009$. Let m and M be the minimal and maximal possible value of $x_1^9 + x_2^9 + \ldots x_n^9$, respectively. Find the $\frac{M}{m}$. Round your answer to nearest integer, if necessary.

8. The real numbers x, y, z, and t satisfy the following equation:

$$2x^2 + 4xy + 3y^2 - 2xz - 2yz + z^2 + 1 = t + \sqrt{y + z - t}$$

Find 100 times the maximum possible value for t.