Algebra A Solutions

1. Find the root that the following three polynomials have in common:

\[ x^3 + 41x^2 - 49x - 2009 \]
\[ x^3 + 5x^2 - 49x - 245 \]
\[ x^3 + 39x^2 - 117x - 1435 \]

(Hint: use all three polynomials.)

Solution. Since the answer to each question on this test is integer-valued, we are looking for integer solutions. It is known that an integer root of a monic polynomial having integer coefficients must divide the constant coefficient. In our case the common root must divide GCD(2009, 1435, 245). It is easy to factor 245 = 5 \times 7^2 and check that 7 divides both 2009 and 1435, but 49 does not divide 435, and 5 does not divide 2009. Hence the GCD is 7, so the common roots (if any) must be \( \pm 1 \) or \( \pm 7 \). One can check that only 7 is a common root.

2. Given that \( P(x) \) is the least degree polynomial with rational coefficients such that

\[ P(\sqrt{2} + \sqrt{3}) = \sqrt{2} \]

find \( P(10) \).

Solution. We compute the first few powers of \( x = \sqrt{2} + \sqrt{3} \) and try to find a relation between them.

\[ x^0 = 1 \]
\[ x^1 = \sqrt{2} + \sqrt{3} \]
\[ x^2 = 5 + 2\sqrt{6} \]
\[ x^3 = 11\sqrt{2} + 9\sqrt{3} \]

At this point, it seems possible that some expression \( ax^3 + bx \) gives us \( \sqrt{2} \). Indeed, requiring the coefficients of both \( \sqrt{2} \) and \( \sqrt{3} \) to be 0 in the expression \( ax^3 + bx - \sqrt{2} = 0 \), where \( a, b \) are rationals, we find \( 11a + b - 1 = 0 \) and \( 9a + b = 0 \), with the unique solution \((a, b) = (1/2, -9/2)\). Hence \( P(x) = x^3/2 - 9x/2 \), and \( P(10) = 455 \). We can justify that \( P(x) \) needs to be of degree at least 3: if it were of degree at most two, then we would have \( ax^2 + bx + c - \sqrt{2} = 0 \) for some rational numbers \( a, b, c \). We need the coefficients of \( \sqrt{2}, \sqrt{3}, \sqrt{6} \) and 1 to be 0 (in more advanced terms: these 4 numbers are linearly independent over \( \mathbb{Q} \)), which imposes 4 equations on our 3 coefficients, and will not have a solution.

3. Let \( x_1, x_2, ... x_{10} \) be non-negative real numbers such that \( \frac{x_1}{1} + \frac{x_2}{2} + ... + \frac{x_{10}}{10} \leq 9 \). Find the maximum possible value of \( \frac{x_1^2}{1} + \frac{x_2^2}{2} + ... + \frac{x_{10}^2}{10} \).

Solution. If we denote \( x_i/i \) by \( a_i \), we must maximize \( \sum ia_i^2 \) subject to the condition \( \sum a_i \leq 9 \). Intuitively, the sum is large if the \( a_i \)-s with large weights are themselves large. Indeed the sum is largest when all weight is placed on \( a_{10} \): \( \sum ia_i^2 \leq 10 \sum a_i^2 \leq 10(\sum a_i)^2 \leq 810 \). We have equality if \( a_{10} = 9 \), and all other numbers are 0.
4. Find the smallest positive $\alpha$ (in degrees) for which all the numbers
\[
\cos \alpha, \cos 2\alpha, \ldots, \cos 2^n \alpha, \ldots
\]
are negative.

**Solution.** 120. The answer can be guessed if one tries a few ‘famous’ values (such as $\pi/4$, $\pi/3$ etc.). The proof requires a bit more work. The cosine function is periodic, hence we can restrict our attention to the interval $[0, 2\pi]$. Also, $\cos \alpha < 0$, so we are in the interval $(\pi/2, 3\pi/2)$. If there is a good $\alpha$ in the interval $(\pi, 3\pi/2)$, then $\alpha/2$ is also good and smaller than $\alpha$. Also, $\alpha \neq \pi$ so we obtained that if there is a smallest $\alpha$, it must satisfy $\pi/2 < \alpha < \pi$.

Generally the condition $\cos 2^n \alpha < 0$ means that there is an integer $l_n$ such that $\pi/2 + 2l_n \pi < 2^n \alpha < 3\pi/2 + 2l_n \pi$, or with the notation $x = 2\alpha/\pi$:
\[
4l_n + 1 < 2^n x < 4l_n + 3
\]
From the inequality on $\alpha$ we have $[x] = 1$ (where $[.]$ denotes the floor function). We prove that in base 4, $x$ can be written as $x = 1.1111 \ldots$. The inequality $4l_n + 1 < 2^n x < 4l_n + 3$ means exactly that $2^n x$ written in base 4 has the digit 1 or 2 on the unit position. Multiplying by an appropriate power of 4, any digit in the expansion of $x$ can be shifted to the unit position - hence every digit of $x$ is 1 or 2. This also holds for $2x$: multiplying by an appropriate power of 2, any digit in the expansion of $2x$ can be shifted to the position of the units, and so every digit of $2x$ is 1 or 2. From this it follow that all digits of $x$ are 1. Suppose the contrary, then $x = 1.1111 \ldots 112 \ldots$ (where there are $m$ 1-s after the dot), and so $4^{m-1} x = 11 \ldots 112 \ldots$.

Multiplying this by 2, the digit that appears on the unit position is 3, since the fractional part is greater than $1/2$. Hence $2^{2m-1} x = 22 \ldots 223.0 \ldots$ or $2^{2m-1} x = 22 \ldots 223.1 \ldots$ and this is a contradiction with the fact that the digit on the unit position of $2^n x$ is 1 or 2. Hence $x = 1.11111 \ldots = 4/3$ and $\alpha = 2\pi/3$.

5. Find the maximal positive integer $n$, so that for any real number $x$ we have $\sin^n x + \cos^n x \geq \frac{1}{n}$.

**Solution.** 8. For $x = \pi$ we need $(-1)^n \geq \frac{1}{n}$, hence $n$ is even. Since $\sin^2 x + \cos^2 x = 1$, and we need to find the minimum of $\sin^n x + \cos^n x = \sin^{2n/2} x + \cos^{2n/2} x$, one would expect that the minimum occurs when $\sin x = \cos x$ (in analogy with the AM-GM or the power mean inequality). For $x = \pi/4$, we have $\sin x = \cos x = 1/\sqrt{2}$ so we need $2 \times \frac{1}{\sqrt{2}} \geq 1/n$ which implies $n \leq 8$. This can be proved from the binomial formula or using calculus. The last step is to prove the inequality for $n = 8$. One way to do this is to use the power mean inequality:
\[
\frac{\left(\sin^8 x + \cos^8 x\right)^{1/2}}{2} \geq \left(\frac{\sin^2 x + \cos^2 x}{2}\right)^{1/2},
\]
hence $\sin^n x + \cos^n x \geq \frac{1}{2}$, which is the desired result. Using calculus, one can give a different solution by finding the minimum of $f(x) = \sin^n x + \cos^n x$.

6. Find the number of functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ for which $f(h+k) + f(hk) = f(h)f(k) + 1$, $\forall h, k \in \mathbb{Z}$.

**Solution.** 3. Putting $(h, k) = (0, 0)$ we get $(f(0) - 1)^2 = 0$ hence $f(0) = 1$. Then let $(h, k) = (1, -1)$ to find $f(0) + f(-1) = f(1)f(-1) + 1$ hence $f(-1)(f(1) - 1) = 0$. So there are two cases: first, if $f(1) = 1$, then letting $(h, k) = (1, k)$ yields $f(1+k) + f(k) = f(1)f(k) + 1$, or
7. Let \( x \) be a real number for all \( k \). On the other hand, if \( x \) is given, then \( f(k) = f(1) \). From the first equation we can express \( f(-2) = 1 - f(1) \) and substitute this into the second one: \( 1 - f(1) = f(1)(1 - f(1)) + 1 \). Hence there are two subcases: \( f(1) = 0 \) or \( 2. \) Recall that \( f(1 + k) + f(k) = f(1)f(k) + 1 \). If \( f(1) = 0 \), this becomes \( f(1 + k) = 1 - f(k) \) and since \( f(0) = 1 \) the function takes the values 0 and 1 alternately: \( f(k) = \frac{1 + (-1)^k}{2} \). On the other hand, if \( f(1) = 2 \), then the equation becomes \( f(1 + k) = f(k) + 1 \), hence \( f(k) = k + 1 \) for all \( k \). We obtained 3 solutions: \( f = 1, f(k) = \frac{1 + (-1)^k}{2} \) and \( f(k) = k + 1 \).

8. The real numbers \( x, y, z, \) and \( t \) satisfy the following equation:

\[
2x^2 + 4xy + 3y^2 - 2xz - 2yz + z^2 + 1 = t + \sqrt{y + z} - t
\]

Find 100 times the maximum possible value for \( t \).

Solution. 125. The equation is equivalent to the following

\[
(z - x - y - 1/2)^2 + (x + y - 1/2)^2 + (y - 1/2)^2 + (\sqrt{y + z} - t - 1/2)^2 = 0
\]

Hence \( y = 1/2, x = 0, z = 1 \) and \( t = 5/4 \).

We would like to give some motivation for this: First, note that you are on the right track once you want to complete squares. The equation is of the form

\[
a = t + \sqrt{b - t}
\]

it is very natural to move everything to one side and write it as

\[
a - b - 1/4 + (b - t) - \sqrt{b - t} + 1/4 = 0
\]

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or also $a - b - 1/4 + (\sqrt{b - t} - 1/2)^2 = 0$. Hence we must examine the expression $a - b - 1/4$ in more detail. In our case this is a non-homogenous polynomial in $x, y, z$, such that each term is of degree at most 2.

$$2x^2 + 4xy + 3y^2 - 2xz - 2yz + z^2 + 3/4 - y - z$$

However, we can actually transform it into a homogenous polynomial using the substitution $x = s/v, y = t/v, z = u/v$ in which case it becomes

$$\frac{2s^2 + 4st + 3t^2 - 2su - 2tu + u^2 + 3/4v^2 - tv - uv}{v^2}$$

The reason to do this is that there is a standard theorem about how one can reduce such homogenous quadratic polynomials, called quadratic forms: The Canonical Form Theorem for Symmetric Quadratic Forms. This provides a standard and straightforward method in which you can diagonalize any symmetric quadratic form - basically by completing squares in the appropriate way. Applying that theorem to our little problem may be a bit of an overkill, but at least it shows some connection of this problem to some classical area of mathematics, making it less sketchy.