Algebra A Solutions

1. Find the root that the following three polynomials have in common:

$$x^3 + 41x^2 - 49x - 2009$$
$$x^3 + 5x^2 - 49x - 245$$
$$x^3 + 39x^2 - 117x - 1435$$

(Hint: use all three polynomials.)

Solution. 7. Since the answer to each question on this test is integer-valued, we are looking for integer solutions. It is known that an integer root of a monic polynomial having integer coefficients must divide the constant coefficient. In our case the common root must divide $GCD(2009, 1435, 245)$. It is easy to factor $245 = 5 \times 7^2$ and check that 7 divides both 2009 and 1435, but 49 does not divide 435, and 5 does not divide 2009. Hence the GCD is 7, so the common roots (if any) must be ± 1 or ± 7. One can check that only 7 is a common root.

2. Given that $P(x)$ is the least degree polynomial with rational coefficients such that

$$P(\sqrt{2} + \sqrt{3}) = \sqrt{2},$$

find $P(10)$.

Solution. 455. We compute the first few powers of $x = \sqrt{2} + \sqrt{3}$ and try to find a relation between them.

$$x^0 = 1$$
$$x^1 = \sqrt{2} + \sqrt{3}$$
$$x^2 = 5 + 2\sqrt{6}$$
$$x^3 = 11\sqrt{2} + 9\sqrt{3}$$

At this point, it seems possible that some expression $ax^3 + bx$ gives us $\sqrt{2}$. Indeed, requiring the coefficients of both $\sqrt{2}$ and $\sqrt{3}$ to be 0 in the expression $ax^3 + bx - \sqrt{2} = 0$, where a, b are rationals, we find $11a + b - 1 = 0$ and $9a + b = 0$, with the unique solution $(a, b) = (1/2, -9/2)$. Hence $P(x) = x^3/2 - 9x/2$,and $P(10) = 455$. We can justify that $P(x)$ needs to be of degree at least 3: if it were of degree at most two, then we would have $ax^2 + bx + c - \sqrt{2} = 0$ for some rational numbers a, b, c. We need the coefficients of $\sqrt{2}$,$\sqrt{3}$,$\sqrt{6}$ and 1 to be 0 (in more advanced terms: these 4 numbers are linearly independent over $\mathbb{Q}$), which imposes 4 equations on our 3 coefficients, and will not have a solution.

3. Let $x_1, x_2, ...x_{10}$ be non-negative real numbers such that $\frac{x_1}{1} + \frac{x_2}{2} + ... + \frac{x_{10}}{10} \leq 9$. Find the maximum possible value of $\frac{{x_1}^2}{1} + \frac{{x_2}^2}{2} + ... + \frac{{x_{10}}^2}{10}$.

Solution. 810. If we denote x_i/i by a_i, we must maximize $\sum i a_i^2$ subject to the condition $\sum a_i \leq 9$. Intuitively, the sum is large if the a_i-s with large weights are themselves large. Indeed the sum is largest when all weight is placed on a_{10}: $\sum i a_i^2 \leq 10 \sum a_i^2 \leq 10(\sum a_i)^2 \leq 810$. We have equality if $a_{10} = 9$, and all other numbers are 0.

4. Find the smallest positive α (in degrees) for which all the numbers

$$\cos\alpha, \cos 2\alpha, \ldots, \cos 2^n\alpha, \ldots$$

are negative.

Solution. 120. The answer can be guessed if one tries a few 'famous' values (such as $\pi/4, \pi/3$ etc.). The proof requires a bit more work. The cosine function is periodic, hence we can restrict our attention to the interval $[0, 2\pi]$. Also, $\cos\alpha < 0$, so we are in the interval $(\pi/2, 3\pi/2)$. If there is a good α in the interval $(\pi, 3\pi/2)$, then $\alpha/2$ is also good and smaller than α. Also, $\alpha \neq \pi$ so we obtained that if there is a smallest α, it must satisfy $\pi/2 < \alpha < \pi$.

Generally the condition $\cos 2^n\alpha < 0$ means that there is an integer l_n such that $\pi/2 + 2l_n\pi < 2^n\alpha < 3\pi/2 + 2l_n\pi$, or with the notation $x = 2\alpha/\pi$:

$$4l_n + 1 < 2^n x < 4l_n + 3$$

From the inequality on α we have $[x] = 1$ (where $[.]$ denotes the floor function). We prove that in base 4, x can be written as $x = 1.11111\ldots$. The inequality $4l_n + 1 < 2^n x < 4l_n + 3$ means exactly that $2^n x$ written in base 4 has the digit 1 or 2 on the unit position. Multiplying by an appropriate power of 4, any digit in the expansion of x can be shifted to the unit position - hence every digit of x is 1 or 2. This also holds for $2x$: multiplying by an appropriate power of 2, any digit in the expansion of $2x$ can be shifted to the position of the units, and so every digit of $2x$ is 1 or 2. From this it follow that all digits of x are 1. Suppose the contrary, then $x = 1.1111\ldots112\ldots$ (where there are m 1-s after the dot), and so $4^{m-1}x = 11\ldots11.2\ldots$. Multiplying this by 2, the digit that appears on the unit position is 3, since the fractional part is greater than 1/2. Hence $2^{2m-1}x = 22\ldots223.0\ldots$ or $2^{2m-1}x = 22\ldots223.1\ldots$ and this is a contradiction with the fact that the digit on the unit position of $2^n x$ is 1 or 2. Hence $x = 1.11111\ldots = 4/3$ and $\alpha = 2\pi/3$.

5. Find the maximal positive integer n, so that for any real number x we have $\sin^n x + \cos^n x \geq \frac{1}{n}$.

Solution. 8. For $x = \pi$ we need $(-1)^n \geq \frac{1}{n}$, hence n is even. Since $\sin^2 x + \cos^2 x = 1$, and we need to find the minimum of $\sin^n x + \cos^n x = \sin^{2\times n/2} x + \cos^{2\times n/2} x$, one would expect that the minimum occurs when $\sin x = \cos x$ (in analogy with the AM-GM or the power mean inequality). For $x = \pi/4$, we have $\sin x = \cos x = 1/\sqrt{2}$ so we need $2 \times \frac{1}{2^{\frac{n}{2}}} \geq 1/n$ which implies $n \leq 8$. This can be proved from the binomial formula or using calculus. The last step is to prove the inequality for $n = 8$. One way to do this is to use the power mean inequality:

$$\left(\frac{\sin^8 x + \cos^8 x}{2}\right)^{\frac{1}{8}} \geq \left(\frac{\sin^2 x + \cos^2 x}{2}\right)^{\frac{1}{2}}$$

hence $\frac{\sin^8 x + \cos^8 x}{2} \geq \frac{1}{2^4}$, which is the desired result. Using calculus, one can give a different solution by finding the minimum of $f(x) = \sin^n x + \cos^n x$.

6. Find the number of functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ for which $f(h+k) + f(hk) = f(h)f(k) + 1$, $\forall\, h, k \in \mathbb{Z}$.

Solution. 3. Putting $(h, k) = (0, 0)$ we get $(f(0) - 1)^2 = 0$ hence $f(0) = 1$. Then let $(h, k) = (1, -1)$ to find $f(0) + f(-1) = f(1)f(-1) + 1$ hence $f(-1)(f(1) - 1) = 0$. So there are two cases: first, if $f(1) = 1$, then letting $(h, k) = (1, k)$ yields $f(1+k) + f(k) = f(1)f(k) + 1$, or also $f(k+1) = 1$, that is f is constant, equal to 1. The second case is $f(-1) = 0$, and then we let $(h, k) = (-1, -1)$ to get $f(-2) + f(1) = 1$, and $(h, k) = (1, -2)$ to get $f(-2) = f(1)f(-2) + 1$. From the first equation we can express $f(-2) = 1 - f(1)$ and substitute this into the second one: $1 - f(1) = f(1)(1 - f(1)) + 1$ or also $(1 - f(1))^2 = 1$. Hence there are two subcases: $f(1)$ is 0 or 2. Recall that $f(1+k) + f(k) = f(1)f(k) + 1$. If $f(1) = 0$, this becomes $f(1+k) = 1 - f(k)$ and since $f(0) = 1$ the function takes the values 0 and 1 alternately: $f(k) = \frac{1+(-1)^k}{2}$. On the other hand, if $f(1) = 2$, then the equation becomes $f(1+k) = f(k) + 1$, hence $f(k) = k + 1$ for all k. We obtained 3 solutions: $f = 1$, $f(k) = \frac{1+(-1)^k}{2}$ and $f(k) = k + 1$.

7. Let $x_1, x_2, \ldots, x_n$ be a sequence of integers, such that $-1 \leq x_i \leq 2$, for $i = 1, 2, \ldots, n$, $x_1 + x_2 + \cdots + x_n = 7$ and $x_1^8 + x_2^8 + \ldots x_n^8 = 2009$. Let m and M be the minimal and maximal possible value of $x_1^9 + x_2^9 + \ldots x_n^9$, respectively. Find $\frac{M}{m}$.

Solution. $\frac{M}{m} = 511$. The x_i-s that are 0 do not matter at all, they can be disregarded. The remaining x_i are -1, 1, or 2 and the only thing that matters is how many of the x_i have each particular value. If $a, b, c \geq 0$ denote the number of -1-s, 1-s, and 2-s, respectively, then $x_1^8 + x_2^8 + \ldots x_n^8 = 2009$ says that $a + b + 256c = 2009$: each x_i which is -1 contributes 1 to the sum, for a total of a, also b is the contribution of the 1-s, and $256c$ is the contribution of the 2-s (each gives 256). Similarly, $x_1 + x_2 + \cdots + x_n = -a + b + 2c$, and $x_1^9 + x_2^9 + \ldots x_n^9 = -a + b + 512c$. Hence given that a, b, c are nonnegative integers such that

$$\begin{aligned} -a + b + 2c &= 7 \\ a + b + 256c &= 2009. \end{aligned}$$

We must find the extremal values of $-a + b + 512c$. We can write $-a + b + 512c = (-a + b + 2c) + 510c = 7 + 510c$, and this expression is extremal when c is extremal. Adding the first two equations we get $2b + 258c = 2016$, and since $b \geq 0$, this implies $c \leq 2016/258 = 7 + 35/43$. Since c is a non-negative integer, $7 \leq 7 + 510c \leq 7 + 510 \times 7$. Also, it is easy to check that if $(a, b) = (1001, 1008)$ then $c = 0$; and if $(a, b) = (112, 105)$, then $c = 7$. So the extrema are 7 and 7×511.

8. The real numbers x, y, z, and t satisfy the following equation:

$$2x^2 + 4xy + 3y^2 - 2xz - 2yz + z^2 + 1 = t + \sqrt{y + z - t}$$

Find 100 times the maximum possible value for t.

Solution. 125. The equation is equivalent to the following

$$(z - x - y - 1/2)^2 + (x + y - 1/2)^2 + (y - 1/2)^2 + \left(\sqrt{y + z - t} - 1/2\right)^2 = 0$$

Hence $y = 1/2$, $x = 0$, $z = 1$ and $t = 5/4$.

We would like to give some motivation for this: First, note that you are on the right track once you want to complete squares. The equation is of the form

$$a = t + \sqrt{b - t}$$

it is very natural to move everything to one side and write it as

$$a - b - 1/4 + (b - t) - \sqrt{b - t} + 1/4 = 0$$

or also $a - b - 1/4 + \left(\sqrt{b-t} - 1/2\right)^2 = 0$. Hence we must examine the expression $a - b - 1/4$ in more detail. In our case this is a non-homogenous polynomial in x, y, z, such that each term is of degree at most 2.

$$2x^2 + 4xy + 3y^2 - 2xz - 2yz + z^2 + 3/4 - y - z$$

However, we can actually transform it into a homogenous polynomial using the substitution $x = s/v$,$y = t/v$,$z = u/v$ in which case it becomes

$$\frac{2s^2 + 4st + 3t^2 - 2su - 2tu + u^2 + 3/4v^2 - tv - uv}{v^2}$$

The reason to do this is that there is a standard theorem about how one can reduce such homogenous quadratic polynomials, called quadratic forms: The Canonical Form Theorem for Symmetric Quadratic Forms. This provides a standard and straightforward method in which you can diagonalize any symmetric quadratic form - basically by completing squares in the appropriate way. Applying that theorem to our little problem may be a bit of an overkill, but at least it shows some connection of this problem to some classical area of mathematics, making it less sketchy.