Number Theory A

1. You are given that

$$17! = 355687ab8096000$$

for some digits a and b. Find the two-digit number $\overline{ab}$ that is missing above.

2. Find the number of ordered pairs (a, b) of positive integers that are solutions of the following equation:

$$a^2 + b^2 = ab(a + b)$$

3. Find the sum of all prime numbers p which satisfy

$$p = a^4 + b^4 + c^4 - 3$$

for some primes (not necessarily distinct) a, b and c.

4. Find the sum of all integers x for which there is an integer y, such that $x^3 - y^3 = xy + 61$.

5. Suppose that for some positive integer n, the first two digits of 5^n and 2^n are identical. Suppose the first two digits are a and b in this order. Find the two-digit number $\overline{ab}$.

6. Let $s(m)$ denote the sum of the digits of the positive integer m. Find the largest positive integer that has no digits equal to zero and satisfies the equation

$$2^{s(n)} = s(n^2)$$

7. Let $S = \{p/q | q \leq 2009, p/q < 1257/2009, p, q \in \mathbb{N}\}$. If the maximum element of S is p_0/q_0 in reduced form, find $p_0 + q_0$.

8. Find the largest positive integer k such that $\phi(\sigma(2^k)) = 2^k$. ($\phi(n)$ denotes the number of positive integers that are smaller than n and relatively prime to n, and $\sigma(n)$ denotes the sum of divisors of n). As a hint, you are given that $641 | 2^{32} + 1$.