PUMaC 2009
The Team Round

November 21, 2009

Rules

1. You have exactly 40 minutes to solve this Team Round. During this time, you and your teammates may work together and collaborate in any way you wish. The round has been designed to encourage and, indeed, necessitate teamwork. So please collaborate and coordinate with each other to solve as much of the round as you can.

2. You have to solve the given crossword. The clues are mathematical problems of varying difficulty levels. Each blank of the crossword can be filled with exactly *one* character. Note especially that we have used the word *character* and not *digit*. Consider this to be a very broad hint about at least one blank in the crossword.

3. The crossword has a *unique* solution.

4. Scoring is as follows: for each blank filled in correctly, you get 0.5 points. Along with that, for every clue you solve correctly, you get an additional number of points as indicated in the parentheses beside the clue itself.

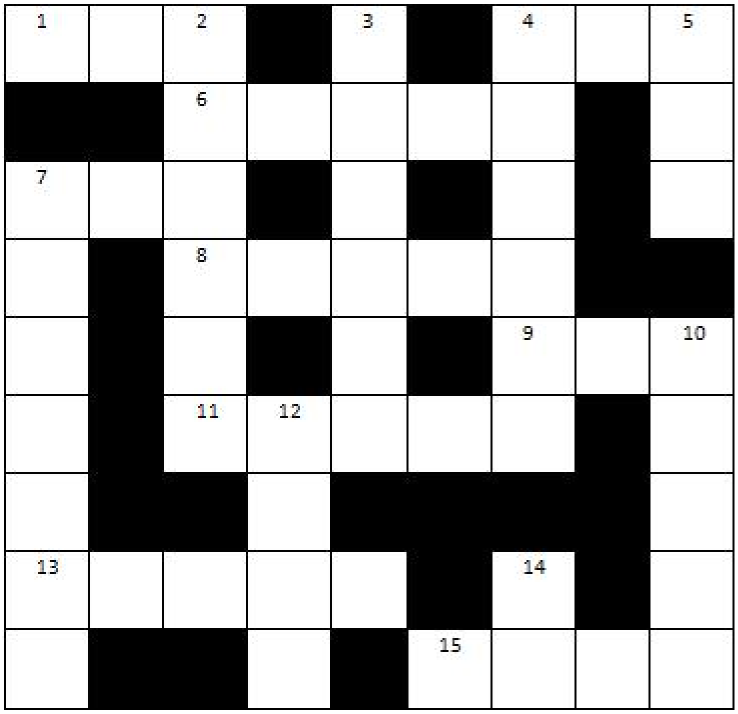
5. You may *NOT* use a calculator for the test. No mathematical program or software may be consulted either. Real mathematics is all about pen and paper, we believe. Not to mention solving crossword puzzles.

6. You do *not* have to show your work. So, if you have some of the characters of a solution from solving other clues, intelligent guessing might come in handy. But keep the time limit in mind.

7. All solutions must be *exact* solutions. Keep in mind that we have not used any approximations.

8. This round is supposed to be a fun round, though it does have a lot of points and can make a difference to your final score. But above all, it is a team game, so we hope you will divide the work fairly and uniformly throughout your team based on your individual strengths. Above all, we hope you have a lot of fun solving this round.

Good luck!



1 Across

- **1 Across** There is only one square number n that can be written in the form $1+p+p^2+p^3+p^4$. It can also be expressed as $k!+1$ for some $k \in \mathbb{N}$. What is the square number n? (3 characters; 2 points)

- **4 Across** For integers $1 \leq k \leq 96$ let $a_k = \frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{96}$. Find $a_1 + a_1^2 + a_2^2 + \ldots + a_{96}^2$. (3 characters; 4 points)

- **6 Across** For a finite set M we define the power set of M to be the set of all subsets of M, and we denote the power set of M by $P(M)$. For a set M with 4 elements, determine the number of functions $f : P(M) \to P(M)$ that satisfy the relation

$$f(A) \cup f(B) = f(A \cup B)$$

for any sets $A, B \in P(M)$. (5 characters; 5 points)

- **7 Across** A line drawn from vertex A of equilateral $\triangle$ABC meets BC at D and the circumcircle at P. If $BP = 55$ and $PC = 220$, find AD. (3 characters; 4 points)

- **8 Across** We define a binary operation on the integers : $x \S y$, such that:

1. $x \S 0 = x$, for all x.
2. $(x+1) \S y + x \S (y+1) = 3(x \S y) - xy + 2y$ for all integers x, y.

Find $178 \S 255$. (5 characters; 5 points)

- **9 Across** Suppose n is the number of 5-digit numbers that can be formed with at least one digit repeated. Write down the *square* of the sum of the digits of n. (3 characters; 3 points)

- **11 Across** The sequence x_n of real numbers is defined by: $x_1 = 100$, $x_2 = 200$, $x_n = \frac{1+x_{n-1}}{x_{n-2}}$ if $n \geq 3$. The number x_{2009} can be expressed as $0.a_1a_2a_3 \ldots a_m$, where the a_i are the digits after the decimal points. Write down the entire string of digits after the decimal point of x_{2009}. In your answer, ignore the decimal point, i.e. if the answer is 0.098765432, then write 098765432. Do not ignore leading zeroes, if any. (5 characters; 5 points)

- **13 Across** There are 10! permutations $s_0s_1 \ldots s_9$ of $0, 1, \ldots, 9$. How many of them satisfy

$$s_k \geq k - 2$$

for $k = 0, 1, \ldots, 9$? (5 characters; 5 points)

- **15 Across** Let C be the unit circle $x^2 + y^2 = 1$. A point P is chosen randomly on the circumference of C, and another point Q is chosen randomly from the interior of C. Both these points are chosen independently and uniformly over their domains. Let R be the rectangle with sides parallel to the x and y-axes with diagonal PQ. Suppose the probability that no point of R lies outside of C is $\frac{1000}{k\pi}$. Find k. (4 characters; 3 points)

2 Down

- **2 Down** There are 9 people standing in a line at a supermarket. What is the number of ways they can stand so that Arthur is ahead of Erick? [Note that standing *ahead* of someone means occupying *any* of the positions in front of him or her.] (6 characters; 3 points)

- **3 Down** A school has 2009 students. The principal asks each student to submit a randomly chosen real number between 0 and 1. She then ranks these numbers in a list of decreasing order, and decides to use the 456th largest number as the fraction of students that are going to get an overall pass this year. What is the expected fraction of students that get a passing grade? Express your answer as a reduced fraction and concatenate the numerator and denominator. (For instance, if you think that the answer is $\frac{1734}{274}$, you would submit 1734274.) (6 characters; 5 points)

- **4 Down** For any $n \in \mathbb{N}$, let $S(n)$ be the sum of the digits of n. Let $M = \max\{S(a)+S(b)+S(c) \mid a+b+c = 2009,\ a,b,c \in \mathbb{N}\}$. How many triples (a,b,c) of natural numbers are there such that $S(a)+S(b)+S(c) = M$? (6 characters; 10 points)

- **5 Down** In right $\triangle ABC$, P and Q are on legs BC and AC, respectively, such that $CP = CQ = 20$. Through the point of intersection, R, of AP and BQ, a line is drawn also passing through C and meeting the hypotenuse AB at S. The extension of PQ meets line AB at T. Suppose $AB = 100$, and $AC = 80$. Then, if the length of TS is k, find k. (3 characters; 5 points)

- **7 Down** The polynomial $p(x)$ is the polynomial of smallest degree which satisifies the following conditions:

  - The coefficients of p are integers.
  - All the roots of p are integers.
  - $p(0) = -1$
  - $p(3) = 128$.

  Write down $p(6)$ in base 3. (7 characters; 6 points)

- **10 Down** The sidelengths of a triangle are 130, 144, and 194. What is the area of its circumcircle? (5 characters; 3 points)

- **12 Down** This number, known as the “Hardy-Ramanujan Number”, is the smallest positive integer that can be expressed as the sum of two cubes in two distinct ways. In other words, it is the *least* $n \in \mathbb{N}$ that satisfies $n = a^3 + b^3 = c^3 + d^3$ for positive integers a, b, c and d, which are all distinct. We give you the following information: two of the numbers a, b, c and d are 1 and 10. (4 characters; 4 points)

- **14 Down** What is one fourth of the maximum number of points you can get for solving this crossword? If strategy fails you on this one, just count. (2 characters; 1 point)