PUMaC 2009

Algebra B

1. If ϕ is the Golden Ratio, we know that $\frac{1}{\phi} = \phi - 1$. Define a new positive real number, called ϕ_d, where $\frac{1}{\phi_d} = \phi_d - d$ (so $\phi = \phi_1$). Given that $\phi_{2009} = \frac{a+\sqrt{b}}{c}$, a, b, c positive integers, and the greatest common divisor of a and c is 1, find $a + b + c$.

2. Let $p(x)$ be the polynomial with leading coefficent 1 and rational coefficents, such that

$$p(\sqrt{3+\sqrt{3+\sqrt{3+\dots}}}) = 0,$$

and with the least degree among all such polynomials. Find $p(5)$.

3. Find the root that the following three polynomials have in common:

$$x^3 + 41x^2 - 49x - 2009$$

$$x^3 + 5x^2 - 49x - 245$$

$$x^3 + 39x^2 - 117x - 1435$$

4. Given that $P(x)$ is the least degree polynomial with rational coefficients such that

$$P(\sqrt{2} + \sqrt{3}) = \sqrt{2},$$

find $P(10)$.

5. Let $x_1, x_2, ...x_{10}$ be non-negative real numbers such that $\frac{x_1}{1} + \frac{x_2}{2} + ... + \frac{x_{10}}{10} \leq 9$. Find the maximum possible value of $\frac{{x_1}^2}{1} + \frac{{x_2}^2}{2} + ... + \frac{{x_{10}}^2}{10}$.

6. Find the smallest positive α (in degrees) for which all the numbers

$$\cos\alpha, \cos 2\alpha, \dots, \cos 2^n\alpha, \dots$$

are negative.

7. Find the maximal positive integer n, so that for any real number x we have $\sin^n x + \cos^n x \geq \frac{1}{n}$.

8. Find the number of functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ for which $f(h + k) + f(hk) = f(h)f(k) + 1$, for all integers h and k.