# PUMaC 2009

## Algebra B Solutions

1. If $\phi$ is the Golden Ratio, we know that $\frac{1}{\phi} = \phi - 1$. I will define a new quantity, called $\phi_d$, where $\frac{1}{\phi_d} = \phi_d - d$ (so $\phi = \phi_1$). Given that $\phi_{2009} = \frac{a+\sqrt{b}}{c}$, $a, b, c$ positive integers, and the greatest common divisor of $a$ and $c$ is 1, find $a + b + c$.

   **Solution.** 4038096. Let $x = \phi_{2009}$. The equation $\frac{1}{x} = x - 2009$ is equivalent to $1 = x^2 - 2009x$ or also $x^2 - 2009x - 1 = 0$. This is a quadratic equation with solutions $x_{1,2} = \frac{2009 \pm \sqrt{2009^2+4}}{2}$. From the statement of the problem it follows that we need to consider the solution with the plus sign. Since GCD$(2009, 2)$=1, the fraction is written in the appropriate way, hence $a + b + c = 2009 + 2009^2 + 4 + 2 = 4038096$.

2. Let $p(x)$ be the polynomial with least degree, leading coefficent 1, rational coefficents, and $p(\sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}) = 0$. Find $p(5)$.

   **Solution.** 17. Let $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}$. Using methods of calculus, one can show that $x$ is a well defined real number. Here, however we do not worry about such questions but rather formally manipulate the expression with the square roots, trying to get some relation between the powers of $x$. We have $x^2 = 3 + \sqrt{3 + \sqrt{3 + \ldots}} = 3 + x$ hence $x$ satisfies the equation $x^2 - x - 3 = 0$. From this it follows that $x$ is not rational (since the roots of the quadratic equation above aren't), and so it does not satisfy any equation of the type $x - n = 0$, with $n$ rational. Hence $p(x)$ is at least of degree 2, so $p(x) = x^2 - x - 3$. Then $p(5) = 25 - 5 - 3 = 17$.

3. Find the root that the following three polynomials have in common:

$$x^3 + 41x^2 - 49x - 2009$$

$$x^3 + 5x^2 - 49x - 245$$

$$x^3 + 39x^2 - 117x - 1435$$

   (Hint: use all three polynomials.)

   **Solution.** 7. Since the answer to each question on this test is integer-valued, we are looking for integer solutions. It is known that an integer root of a monic polynomial having integer coefficients must divide the constant coefficient. In our case the common root must divide $GCD(2009, 1435, 245)$. It is easy to factor $245 = 5 \times 7^2$ and check that 7 divides both 2009 and 1435, but 49 does not divide 435, and 5 does not divide 2009. Hence the $GCD$ is 7, so the common roots (if any) must be $\pm 1$ or $\pm 7$. One can check that only 7 is a common root.

4. Given that $P(x)$ is the least degree polynomial with rational coefficients such that

$$P(\sqrt{2} + \sqrt{3}) = \sqrt{2},$$

   find $P(10)$.

   **Solution.** 455. We compute the first few powers of $x = \sqrt{2} + \sqrt{3}$ and try to find a relation between them.

$$x^0 = 1$$

$$x^1 = \sqrt{2} + \sqrt{3}$$
$$x^2 = 5 + 2\sqrt{6}$$
$$x^3 = 11\sqrt{2} + 9\sqrt{3}$$

At this point, it seems possible that some expression $ax^3 + bx$ gives us $\sqrt{2}$. Indeed, requiring the coefficients of both $\sqrt{2}$ and $\sqrt{3}$ to be 0 in the expression $ax^3 + bx - \sqrt{2} = 0$, where $a, b$ are rationals, we find $11a + b - 1 = 0$ and $9a + b = 0$, with the unique solution $(a, b) = (1/2, -9/2)$. Hence $P(x) = x^3/2 - 9x/2$,and $P(10) = 455$. We can justify that $P(x)$ needs to be of degree at least 3: if it were of degree at most two, then we would have $ax^2 + bx + c - \sqrt{2} = 0$ for some rational numbers $a, b, c$. We need the coefficients of $\sqrt{2}$,$\sqrt{3}$,$\sqrt{6}$ and 1 to be 0 (in more advanced terms: these 4 numbers are linearly independent over $\mathbb{Q}$), which imposes 4 equations on our 3 coefficients, and will not have a solution.

5. Let $x_1, x_2, ...x_{10}$ be non-negative real numbers such that $\frac{x_1}{1} + \frac{x_2}{2} + ... + \frac{x_{10}}{10} \leq 9$. Find the maximum possible value of $\frac{{x_1}^2}{1} + \frac{{x_2}^2}{2} + ... + \frac{{x_{10}}^2}{10}$.

   **Solution.** 810. If we denote $x_i/i$ by $a_i$, we must maximize $\sum ia_i^2$ subject to the condition $\sum a_i \leq 9$. Intuitively, the sum is large if the $a_i$-s with large weights are themselves large. Indeed the sum is largest when all weight is placed on $a_{10}$: $\sum ia_i^2 \leq 10 \sum a_i^2 \leq 10(\sum a_i)^2 \leq 810$. We have equality if $a_{10} = 9$, and all other numbers are 0.

6. Find the smallest positive $\alpha$ (in degrees) for which all the numbers

$$\cos\alpha, \cos 2\alpha, \ldots, \cos 2^n\alpha, \ldots$$

   are negative.

   **Solution.** 120. The answer can be guessed if one tries a few 'famous' values (such as $\pi/4, \pi/3$ etc.). The proof requires a bit more work. The cosine function is periodic, hence we can restrict our attention to the interval $[0, 2\pi]$. Also, $\cos\alpha < 0$, so we are in the interval $(\pi/2, 3\pi/2)$. If there is a good $\alpha$ in the interval $(\pi, 3\pi/2)$, then $\alpha/2$ is also good and smaller than $\alpha$. Also, $\alpha \neq \pi$ so we obtained that if there is a smallest $\alpha$, it must satisfy $\pi/2 < \alpha < \pi$.

   Generally the condition $\cos 2^n\alpha < 0$ means that there is an integer $l_n$ such that $\pi/2 + 2l_n\pi < 2^n\alpha < 3\pi/2 + 2l_n\pi$, or with the notation $x = 2\alpha/\pi$:

$$4l_n + 1 < 2^n x < 4l_n + 3$$

   From the inequality on $\alpha$ we have $[x] = 1$ (where $[.]$ denotes the floor function). We prove that in base 4, $x$ can be written as $x = 1.11111\ldots$. The inequality $4l_n + 1 < 2^n x < 4l_n + 3$ means exactly that $2^n x$ written in base 4 has the digit 1 or 2 on the unit position. Multiplying by an appropriate power of 4, any digit in the expansion of $x$ can be shifted to the unit position - hence every digit of $x$ is 1 or 2. This also holds for $2x$: multiplying by an appropriate power of 2, any digit in the expansion of $2x$ can be shifted to the position of the units, and so every digit of $2x$ is 1 or 2. From this it follow that all digits of $x$ are 1. Suppose the contrary, then $x = 1.1111\ldots112\ldots$ (where there are $m$ 1-s after the dot), and so $4^{m-1}x = 11\ldots11.2\ldots$. Multiplying this by 2, the digit that appears on the unit position is 3, since the fractional part is greater than 1/2. Hence $2^{2m-1}x = 22\ldots223.0\ldots$ or $2^{2m-1}x = 22\ldots223.1\ldots$ and this is a contradiction with the fact that the digit on the unit position of $2^n x$ is 1 or 2. Hence $x = 1.11111\ldots = 4/3$ and $\alpha = 2\pi/3$.

7. Find the maximal positive integer $n$, so that for any real number $x$ we have $\sin^n x+\cos^n x \geq \frac{1}{n}$.

   **Solution.** 8. For $x = \pi$ we need $(-1)^n \geq \frac{1}{n}$, hence $n$ is even. Since $\sin^2 x + \cos^2 x = 1$, and we need to find the minimum of $\sin^n x + \cos^n x = \sin^{2\times n/2} x + \cos^{2\times n/2} x$, one would expect that the minimum occurs when $\sin x = \cos x$ (in analogy with the AM-GM or the power mean inequality). For $x = \pi/4$, we have $\sin x = \cos x = 1/\sqrt{2}$ so we need $2 \times \frac{1}{2^{\frac{n}{2}}} \geq 1/n$ which implies $n \leq 8$. This can be proved from the binomial formula or using calculus. The last step is to prove the inequality for $n = 8$. One way to do this is to use the power mean inequality:

$$(\frac{\sin^8 x + \cos^8 x}{2})^{\frac{1}{8}} \geq (\frac{\sin^2 x + \cos^2 x}{2})^{\frac{1}{2}}$$

   hence $\frac{\sin^8 x+\cos^8 x}{2} \geq \frac{1}{2^4}$, which is the desired result. Using calculus, one can give a different solution by finding the minimum of $f(x)$=$\sin^n x + \cos^n x$.

8. Find the number of functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ for which $f(h+k)+f(hk) = f(h)f(k)+1$, $\forall\ h, k \in \mathbb{Z}$.

   **Solution.** 3. Putting $(h, k) = (0, 0)$ we get $(f(0) - 1)^2 = 0$ hence $f(0) = 1$. Then let $(h, k) = (1, -1)$ to find $f(0) + f(-1) = f(1)f(-1) + 1$ hence $f(-1)(f(1) - 1) = 0$. So there are two cases: first, if $f(1) = 1$, then letting $(h, k) = (1, k)$ yields $f(1+k)+f(k) = f(1)f(k)+1$, or also $f(k+1) = 1$, that is $f$ is constant, equal to 1. The second case is $f(-1) = 0$, and then we let $(h, k) = (-1, -1)$ to get $f(-2) + f(1) = 1$, and $(h, k) = (1, -2)$ to get $f(-2) = f(1)f(-2) + 1$. From the first equation we can express $f(-2) = 1 - f(1)$ and substitute this into the second one: $1-f(1) = f(1)(1-f(1))+1$ or also $(1 - f(1))^2 = 1$. Hence there are two subcases: $f(1)$ is 0 or 2. Recall that $f(1+k)+f(k) = f(1)f(k)+1$. If $f(1) = 0$, this becomes $f(1+k) = 1-f(k)$ and since $f(0) = 1$ the function takes the values 0 and 1 alternately: $f(k) = \frac{1+(-1)^k}{2}$. On the other hand, if $f(1) = 2$, then the equation becomes $f(1 + k) = f(k) + 1$, hence $f(k) = k + 1$ for all $k$. We obtained 3 solutions: $f = 1$, $f(k) = \frac{1+(-1)^k}{2}$ and $f(k) = k + 1$.