PUMaC 2009

Number Theory B

1. Find the number of pairs of integers x and y such that $x^2 + xy + y^2 = 28$.
2. Suppose you are given that for some positive integer n, $1! + 2! + ... + n!$ is a perfect square. Find the sum of all possible values of n.
3. You are given that

$$17! = 355687ab8096000$$

for some digits a and b. Find the two-digit number $\overline{ab}$ that is missing above.

4. Find the number of ordered pairs (a, b) of positive integers that are solutions of the following equation:

$$a^2 + b^2 = ab(a + b)$$

5. Find the sum of all prime numbers p which satisfy

$$p = a^4 + b^4 + c^4 - 3$$

for some primes (not necessarily distinct) a, b and c.

6. Find the sum of all integers x for which there is an integer y, such that $x^3 - y^3 = xy + 61$.
7. Suppose that for some positive integer n, the first two digits of 5^n and 2^n are identical. Suppose the first two digits are a and b in this order. Find the two-digit number $\overline{ab}$.
8. Let $s(m)$ denote the sum of the digits of the positive integer m. Find the largest positive integer that has no digits equal to zero and satisfies the equation

$$2^{s(n)} = s(n^2)$$