PUMaC 2009-10 Power Test

A Version

21 Problems; 86 Points

1 Instructions / Rules

These rules supersede any rules appearing elsewhere about the Power Test.

The problems on this test all require you to prove your answer. This means that you should justify your steps in a way that is correct, complete, and succinct. Credit will not be given for an answer without proof.

For the solution of any given problem, you are allowed to use *without proof* the results of all *previous problems* (that is, problems that appear above in the test), even if your team has not solved those problems.

For a YES/NO question, just answsering YES or NO is worth **ZERO** points. You must provide justification to recieve **ANY** credit.

For problems which ask you to perform a calculation, you must show your work as in any other proof. This work should be clearly justified, not just a sheet of unexplained calculations.

It is not necessary to do the problems in order, though it is a good idea to read all the problems, so that you know what is permissible to assume when doing a given problem. However, please collate the solutions in order in your solution packet.

Consulting printed and online references, as well as the use of calculators, computer programs, Mathematica, etc. is allowed. However you must not discuss the problems with anyone outside of your team.

2 Lattices (4 Problems; 15 Points)

The set of integers is denoted by the symbol $\mathbb{Z}$; similarly, $\mathbb{Z}^n$ denotes the set of ordered n-tuples of integers. If $\vec{a}, \vec{b} \in \mathbb{Z}^n$ are two elements, then $\vec{a} + \vec{b}$ denotes their coordinatewise sum.

For example, $(2, 4, 5) \in \mathbb{Z}^3$, $(54, 2, -3432, 0) \in \mathbb{Z}^4$, $(3, 0, -1) \in \mathbb{Z}^3$, and $(2, 4, 5) + (3, 0, -1) = (5, 4, 4)$.

Definition 2.1. A *lattice in dimension* n is a subset L of $\mathbb{Z}^n$ with the following properties:

1. $(0, \dots, 0) \in L$.
2. If $\vec{a} \in L$ then $-\vec{a} \in L$.

3. If $\vec{a}, \vec{b} \in L$ then $\vec{a} + \vec{b} \in L$.

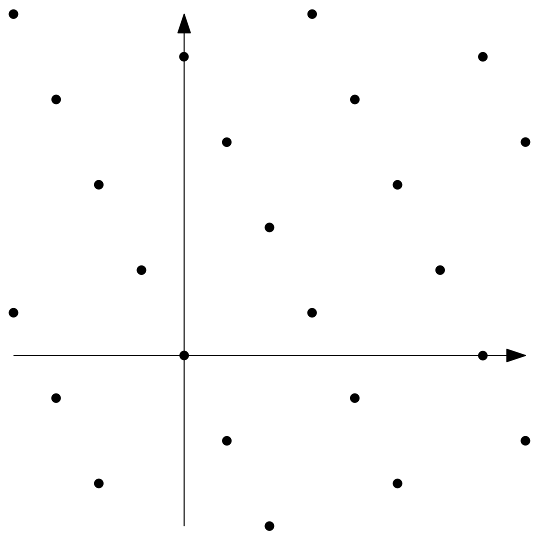

Figure 1: An example of a lattice.

Given a set S of vectors in $\mathbb{Z}^n$, we can form a lattice in dimension n by taking the set of all integer linear combinations of vectors in S. This is called the *lattice generated by* S. For example, the lattice generated by $S = \{(1,1),(1,-1)\}$ is the set of all integer linear combinations of $(1,1)$ and $(1,-1)$, i.e. the subset $(a+b, a-b)$ of $\mathbb{Z}^2$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. This comes out to be simply the set of all $(x,y) \in \mathbb{Z}^2$ with the property that $x+y$ is even. The diagram below shows the points representing the vectors of this lattice.

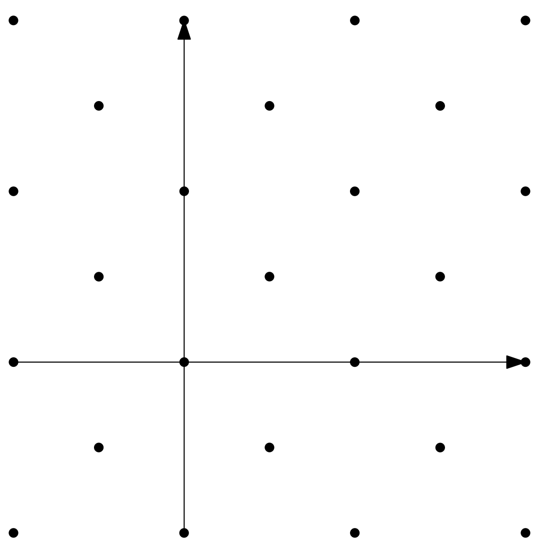

Figure 2: The lattice generated by $\{(1,1),(1,-1)\}$.

Problem 2.1 (3pts). *What are all the lattices in dimension one? (That is, specify their general form in the simplest possible terms.)*

Definition 2.2. A lattice L in dimension n is said to be *full* iff for every $\vec{a} \in \mathbb{Z}^n$, there is a positive integer N so that $N\vec{a} \in L$.

For example, the lattice generated by $\{(1,1),(1,-1)\}$ is full. This is because, if $(a,b) \in \mathbb{Z}^2$, then $2(a,b) = (2a,2b) = ((a+b)+(a-b),(a+b)-(a-b)) = (a+b)(1,1)+(a-b)(1,-1)$. However, the lattice generated by $\{(6,12),(10,20)\}$ is not full. If you draw this then you will see why.

Problem 2.2 (4pts). *Prove that the lattice in dimension n generated by a set S is full if and only if every vector in $\mathbb{Z}^n$ is expressible as a* rational *linear combination of vectors in S, i.e. iff every $\vec{a} \in \mathbb{Z}^n$ is expressible in the form $\sum a_i x_i$, where $x_i \in S$ and $a_i \in \mathbb{Q}$.*

Problem 2.3 (4pts). *Is the lattice generated by $\{(2,1,6),(5,6,8),(1,1,2)\}$ full?*

Problem 2.4 (4pts). *Is the lattice generated by $\{(2,1,6),(5,6,8),(1,2,2)\}$ full?*

3 Determinant and Divisor (5 Problems; 17 Points)

Definition 3.1. Fix a lattice L in $\mathbb{Z}^n$. A *colattice* $\vec{a} + L$ for $\vec{a} \in \mathbb{Z}^n$ is the set of vectors $\{\vec{a} + \vec{\ell}\}$ where $\vec{\ell}$ ranges over all the vectors in L.

Problem 3.1 (2pts). *Show that $\vec{a} + L = \vec{b} + L$ if and only if $\vec{a} - \vec{b} \in L$.*

[Note: two sets A and B are said to be *equal* if they have exactly the same elements, that is $A \subseteq B$ and $B \subseteq A$.]

Problem 3.2 (2pts). *Show that if $\vec{a} - \vec{b} \notin L$, then $(\vec{a} + L) \cap (\vec{b} + L) = \varnothing$.*

[Note: it *might* help to remember that $A \implies B$ is equivalent to proving $\bar{B} \implies \bar{A}$, where the bars stand for the negations of the corresponding statements.]

Definition 3.2. The *determinant* of L, $\Delta = \det L$, is the number of distinct colattices of L.

We remark that the determinant may be infinite. For example, the lattice generated by $\{(2,3),(12,18)\}$ has an infinite determinant.

Problem 3.3 (3pts). *Let L be the lattice generated by $\{(1,2),(2,1)\}$. Draw the colattices $(0,0)+L$, $(0,1)+L$, and $(0,2)+L$. If you drew the diagram correctly, it should sort of jump out that these are distinct colattices. If it doesn't jump out, check your diagram! Now prove that these colattices are in fact distinct without using the diagram. Also, prove that L has no more colattices. Hence, conclude that* $\det L = 3$.

Problem 3.4 (5pts). *Prove that a lattice is full if and only if its determinant is finite.*

Definition 3.3. The *divisor* of a lattice L is the greatest positive integer $d = \operatorname{div} L$ with the property that for every $\vec{a} \in L$, $\vec{a}/d \in \mathbb{Z}^n$.

Problem 3.5 (5pts). *Prove that if L is a full lattice in dimension n, then its determinant is divisible by* $(\operatorname{div} L)^n$.

4 Finite Generation (3 Problems; 13 Points)

Problem 4.1 (3pts)**.** *Prove that if* $L_1 \supsetneqq L_2$*, then* $\det L_1 < \det L_2$ *(or both are* ∞*).*

Definition 4.1. A lattice is said to be *finitely generated* iff it is equal to the lattice generated by a finite set S.

Problem 4.2 (5pts)**.** *Prove that every full lattice has a full sublattice that is finitely generated.*

Problem 4.3 (5pts)**.** *Prove that every full lattice is finitely generated.*

5 Isomorphism Types of Lattices (7 Problems; 28 Points)

Definition 5.1. Two lattices L_1 and L_2 in $\mathbb{Z}^n$ are said to be *isomorphic* iff there exists a linear bijection $f : \mathbb{Z}^n \to \mathbb{Z}^n$ which is also a bijection from L_1 to L_2. [A map $f : \mathbb{Z}^n \to \mathbb{Z}^m$ is said to be *linear* iff $f(\vec{0}) = \vec{0}$, and $f(\vec{a} + \vec{b}) = f(\vec{a}) + f(\vec{b})$]. [A map $f : X \to Y$ is a *bijection* if it gives a one-to-one correspondence between elements of X and elements of Y].

For example, the linear bijection $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by $f(x, y) = (5x + 2y, 2x + y)$ is also a bijection from the lattice L_1 generated by $\{(2, 1), (1, 2)\}$ to the lattice L_2 generated by $\{f(2, 1), f(1, 2)\} = \{(12, 5), (9, 4)\}$; hence L_1 and L_2 are isomorphic. Note that L_2 is also the lattice generated by $\{(3, 0), (0, 1)\}$.

Problem 5.1 (2pts)**.** *Prove that* div *is an isomorphism invariant.*

[Note: This is equivalent to proving that two isomorphic lattices have the same divisor. The next problem is similar.]

Problem 5.2 (2pts)**.** *Prove that* det *is an isomorphism invariant.*

Problem 5.3 (3pts)**.** *Are the lattices generated by* $\{(3, 0), (0, 5)\}$ *and* $\{(1, 0), (0, 15)\}$ *isomorphic?*

Problem 5.4 (3pts)**.** *Are the lattices generated by* $\{(2, 0), (0, 4)\}$ *and* $\{(1, 0), (0, 8)\}$ *isomorphic?*

Problem 5.5 (4pts)**.** *For any two integers* $d \geq 1$ *and* $\Delta \geq 1$ *with* Δ *divisible by* d^2*, give an example of a lattice in* $\mathbb{Z}^2$ *with divisor* d *and determinant* Δ*.*

Problem 5.6 (7pts)**.** *Prove that if two full lattices in* $\mathbb{Z}^2$ *have the same determinant and same divisor then they are isomorphic. Conclude that all full lattices in* $\mathbb{Z}^2$ *are isomorphic to one of the lattices from problem 5.5 (don't forget the result of problem 3.5).*

Problem 5.7 (7pts)**.** *Prove that divisor and determinant do not characterize lattices in dimension three. That is, construct two lattices* L_1 *and* L_2 *in* $\mathbb{Z}^3$ *which have the same determinant and the same divisor but which are not isomorphic.*

6 Canonical Form (2 Problems; 13 Points)

The following theorem is true (proving it is not part of this test).

Theorem 6.1. *Every lattice in dimension* n *is isomorphic to the lattice generated by*

$$\{d_1\vec{e}_1, \ldots, d_n\vec{e}_n\} \tag{6.1}$$

for some integers $d_i \geq 0$ *where* d_i *divides* d_{i+1}. *Furthermore, the sequence of integers* $(d_1; \ldots; d_n)$ *is isomorphism invariant; it is called the* signature *of the lattice.*

Here $\vec{e}_i$ *is the vector* $(0, \ldots, 0, 1, 0, \ldots, 0)$ *where the* 1 *appears in the* i*th entry.*

You may assume it is true for any of your work on problems appearing after this point in the test.

Problem 6.1 (5pts)**.** *Calculate the signature of the lattice generated by:*

$$\{(2, 2, 0), (0, 3, 3)\} \tag{6.2}$$

Problem 6.2 (8pts)**.** *Calculate the signature of the lattice generated by:*

$$\{(0, 2, 5, 3), (5, 4, 5, 7), (5, 9, 7, 1), (5, 7, 5, 7)\} \tag{6.3}$$