PUMaC 2010

Algebra A

1. Find the sum of the coefficients of the polynomial $(63x - 61)^4$.
2. Calculate $\sum_{n=1}^{\infty} \left(\lfloor \sqrt[n]{2010} \rfloor - 1 \right)$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x.
3. Let S be the sum of all real x such that $4^x = x^4$. Find the nearest integer to S.
4. Define $f(x) = x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$. Find the smallest integer x such that $f(x) \geq 50\sqrt{x}$.
5. Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots of f are given by α, β, and γ, find
$$\left(\frac{1}{\alpha - 2}\right)^2 + \left(\frac{1}{\beta - 2}\right)^2 + \left(\frac{1}{\gamma - 2}\right)^2.$$
6. Assume that $f(a + b) = f(a) + f(b) + ab$, and that $f(75) - f(51) = 1230$. Find $f(100)$.
7. The expression $\sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ$ is equal to $p\sqrt{5}/2^{50}$, where p is an integer. Find p.
8. Let p be a polynomial with integer coefficients such that $p(15) = 6$, $p(22) = 1196$, and $p(35) = 26$. Find an integer n such that $p(n) = n + 82$.