1. Find the sum of the coefficients of the polynomial \((63x - 61)^4\).
Solution: 16. The sum of the coefficients of \(f(x) = (63x - 61)^4\) is \(f(1) = (63 - 61)^4 = 16\).

2. Calculate \(\sum_{n=1}^{\infty} \left( \lfloor \sqrt[2010]{n} \rfloor - 1 \right)\).
Solution: 2077. Just calculate it; note that for \(n > 10\) we have \(2^{10} > 2010\) so all those terms are zero. We get \(2009 + 43 + 11 + 5 + 3 + 2 + 1 + 1 + 1 + 1 = 2077\).

3. Find the nearest integer to the sum of all \(x\) where \(4^x = x^4\).
Solution: 5. We immediately see two solutions, 2 and 4, and that there can be no more positive roots. There must be a negative root, however: let \(f(x) = 4^x\) and \(g(x) = x^4\); then \(g(0) = 0\) and \(f(0) = 1\), but \(g\) goes off to infinity as \(x \to -\infty\) and \(f\) goes to 0 as \(x \to \infty\). Plugging in \(x = -1\), we have \(f(-1) = 1/4\) and \(g(x) = 1\); plugging in \(x = -1/2\) we have \(f(-1/2) = 1/2\) and \(g(x) = 1/16\). Therefore the root is between \(-1/2\) and \(-1\), and the nearest integer to the sum of the roots must be 5.

4. Define \(f(x) = x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}\). Find the smallest integral \(x\) such that \(f(x) \geq 50\sqrt{x}\).
Solution: 2400. Noting that \((f(x) - x)^2 = f(x)\), we can solve the quadratic equation for \(f(x)\) to get that
\[
f(x) = x + \frac{1}{2} \pm \sqrt{x + \frac{1}{4}}.
\]
We clearly have to take the positive root (we can notice, for example, that \(f(1) > 1\)). The problem therefore reduces to finding the smallest integral \(x\) such that
\[
x + \frac{1}{2} + \sqrt{x + \frac{1}{4}} \geq 50\sqrt{x}.
\]
It is simple to note that \(x\) has to be fairly large for this to be satisfied (after trying the trivial \(x = 1\)). For large \(x\), \(\sqrt{x + \frac{1}{4}}\) is very, very close to \(\sqrt{x}\), so we can rewrite this as
\[
x + \frac{1}{2} \geq 49\sqrt{x}.
\]
The above is again rewritten as
\[
x^2 - 2400x + \frac{1}{4} \geq 0.
\]
The smallest integer \(x\) satisfying the above is obviously 2400, and since the margin of error here is \(\frac{1}{4}\), our previous approximation is justified.
5. Let \( f(x) = 3x^3 - 5x^2 + 2x - 6 \). If the roots of \( f \) are given by \( \alpha, \beta, \) and \( \gamma, \) find

\[
\left( \frac{1}{\alpha - 2} \right)^2 + \left( \frac{1}{\beta - 2} \right)^2 + \left( \frac{1}{\gamma - 2} \right)^2.
\]

Solution: 68. A polynomial with roots \( \alpha - 2, \beta - 2, \) and \( \gamma - 2 \) is given by

\[ g(x) = f(x + 2) = 3x^3 + 13x^2 + 18x + 2. \]

A polynomial with roots \( 1/(\alpha - 2), 1/(\beta - 2), \) and \( 1/(\gamma - 2) \) is given by

\[ h(x) = 2x^3 + 18x^2 + 13x + 3. \]

Since \( a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca), \) we can find the result by finding the elementary symmetric polynomials on the roots. Here, we have

\[
\frac{1}{\alpha - 2} + \frac{1}{\beta - 2} + \frac{1}{\gamma - 2} = -9
\]

and

\[
\frac{1}{\alpha - 2} \cdot \frac{1}{\beta - 2} \cdot \frac{1}{\gamma - 2} = -\frac{13}{2},
\]

so the desired sum is \((-9)^2 - 2 \cdot \frac{13}{2} = 81 - 13 = 68.\]

6. Assume that \( f(a + b) = f(a) + f(b) + ab, \) and that \( f(75) - f(51) = 1230. \) Find \( f(100). \)

Solution: One has that \( f(0) = f(0) + f(0), \) so \( f(0) = 0. \) Moreover, \( f(n + 1) = f(n) + f(1) + n, \)

so that \( f(n) = \sum_{i=0}^{n-1}(i + f(1)) = \frac{n(n-1)}{2} + nf(1). \) Plugging in \( n = 75 \) and \( n = 51, \) one gets

\[ 2775 - 1275 + 24f(1) = 1230, \]

so \( f(1) = -45. \) Thus, \( f(100) = 3825. \)

7. The expression \( \sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ \) is equal to \( p\sqrt{5}/2^{50}, \) where \( p \) is an integer. Find \( p. \)

Solution: 192. Let \( \omega \) be the root of unity \( e^{2\pi i/90}, \) so we have

\[ \prod_{n=1}^{45} \sin(2n^\circ) = \sum_{n=1}^{45} \frac{\omega^n - 1}{2i\omega^n}. \]

By the symmetry of the sine (and the fact that \( \sin(90^\circ) = 1),

\[ \prod_{n=1}^{45} \sin(2n^\circ) = \prod_{n=46}^{89} \sin(2n^\circ), \]

so

\[ \left| \prod_{n=1}^{45} \sin(2n^\circ) \right|^2 = \sum_{n=1}^{89} \frac{|\omega^n - 1|}{2} = \frac{90}{289}, \]

where we’ve used the usual geometric series sum for roots of unity. The product is clearly positive and real, so it is equal to

\[ \frac{\sqrt{45}}{2^{44}} = \frac{3\sqrt{5}}{2^{44}}, \]

implying that \( p = 3 \cdot 2^6 = 192. \)
8. Let $p$ be a polynomial with integer coefficients such that $p(15) = 6$, $p(22) = 1196$, and $p(35) = 26$. Assume that $p(n) = n + 82$ for some integer $n$. Find $n$.

Solution: 28. Since $p(n) - n - 82 = 0$, the polynomial $p(x) - x - 82$ must factor to $(x - n)q(x)$, where $q(x)$ is another polynomial. The polynomial $q$ will have integer coefficients, because $p(x) - x - 82 = p(x) - p(n) + n - x$, so if we let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_jx^j$, we get

$$p(x) - x - 82 = (a_1 - 1)(x - n) + a_2(x^2 - n^2) + \cdots + a_j(x^j - n^j).$$

Dividing through by $x - n$ clearly leaves a polynomial with integer coefficients, since $x^i - n^i$ is always divisible by $x - n$. In particular, therefore, $q(15)$ and $q(35)$ are integers, so plugging in 15 and 35 we get that $15 - n$ is divisible by $p(15) - 15 - 82 = -91$ and $35 - n$ is divisible by $p(35) - 35 - 82 = -91$. Since the factors of 91 are just $\pm 1$, $\pm 7$, and $\pm 13$, we must have either $15 - n = -13 \implies n = 28$ or $15 - n = -7 \implies n = 22$. The latter case is ruled out because $p(22) = 116 \neq 22 + 82$. 


