PUMaC 2010

Combinatorics A

1. PUMaCDonalds, a newly-opened fast food restaurant, has 5 menu items. If the first 4 customers each choose one menu item at random, the probability that the 4th customer orders a previously unordered item is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2. Let $\underline{xyz}$ represent the three-digit number with hundreds digit x, tens digit y, and units digit z, and similarly let $\underline{yz}$ represent the two-digit number with tens digit y and units digit z. How many three-digit numbers $\underline{abc}$, none of whose digits are 0, are there such that $\underline{ab} > \underline{bc} > \underline{ca}$?

3. Sterling draws 6 circles on the plane, which divide the plane into regions (including the unbounded region). What is the maximum number of resulting regions?

4. Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares, and there are \$5 bills in the top right and bottom left squares, as shown below.

\$1				\$5
	E			
\$5				\$1

   Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m + n$.

5. We say that a rook is "attacking" another rook on a chessboard if the two rooks are in the same row or column of the chessboard and there is no piece directly between them. Let n be the maximum number of rooks that can be placed on a 6×6 chessboard such that each rook is attacking at most one other. How many ways can n rooks be placed on a 6×6 chessboard such that each rook is attacking at most one other?

6. All the diagonals of a regular decagon are drawn. A regular decagon satisfies the property that if three diagonals concur, then one of the three diagonals is a diameter of the circumcircle of the decagon. How many distinct intersection points of diagonals are in the interior of the decagon?

7. Matt is asked to write the numbers from 1 to 10 in order, but he forgets how to count. He writes a permutation of the numbers $\{1, 2, 3 \ldots, 10\}$ across his paper such that:

   (a) The leftmost number is 1.

   (b) The rightmost number is 10.

   (c) Exactly one number (not including 1 or 10) is less than both the number to its immediate left and the number to its immediate right.

   How many such permutations are there?

8. Let N be the sum of all binomial coefficients $\binom{a}{b}$ such that a and b are nonnegative integers and $a + b$ is an even integer less than 100. Find the remainder when N is divided by 144. (Note: $\binom{a}{b} = 0$ if $a < b$, and $\binom{0}{0} = 1$.)