PUMaC 2010

Combinatorics A Solutions

1. PUMaCDonalds, a newly-opened fast food restaurant, has 5 menu items. If the first 4 customers each choose one menu item at random, the probability that the 4th customer orders a previously unordered item is m/n, where m and n are relatively prime positive integers. Find $m+n$.

   Answer: $\boxed{189}$

   Solution: Number the menu items 1 through 5. Without loss of generality, assume the 4th customer orders menu item 1. Then the desired probability is the probability that each of the first 3 customers do not order menu item 1, which is $(4/5)^3 = 64/125$. The answer is $64 + 125 = \boxed{189}$.

2. Let $\underline{xyz}$ represent the three-digit number with hundreds digit x, tens digit y, and units digit z, and similarly let $\underline{yz}$ represent the two-digit number with tens digit y and units digit z. How many three-digit numbers $\underline{abc}$, none of whose digits are 0, are there such that $\underline{ab} > \underline{bc} > \underline{ca}$?

   Answer: $\boxed{120}$

   Solution: If one two-digit number is greater than another, then the tens digit of the first number must be greater than or equal to the tens digit of the second number. Therefore, if $\underline{abc}$ satisfies the given condition, then $a \geq b \geq c$. Now note that if $b = c$, then since $c \leq a$, we have $\underline{bc} \leq \underline{ca}$, a contradiction. Therefore, $a \geq b > c$. Conversely, if $a \geq b > c$, then we can easily see that $\underline{abc}$ satisfies the given condition. Therefore, the problem is equivalent to finding the number of ordered pairs (a, b, c) of integers between 1 and 9 such that $a \geq b > c$.

   If $a = b > c$, we can produce all such (a, b, c) by choosing 2 of the integers between 1 and 9, and setting a and b equal to the larger integer, and c equal to the smaller integer. If $a > b > c$, we can produce all such (a, b, c) by choosing 3 of the integers between 1 and 9, and setting a equal to the largest integer, b equal to the middle integer, and c equal to the smallest integer. Therefore the answer is $\binom{9}{2} + \binom{9}{3} = 45 + 120 = \boxed{120}$.

3. Sterling draws 6 circles on the plane, which divide the plane into regions (including the unbounded region). What is the maximum number of resulting regions?

   Answer: $\boxed{32}$

   Solution: Let $R(n)$ be the greatest number of regions that n circles can divide the plane into. We want to calculate $R(n+1)$ in terms of $R(n)$.

   Suppose we have drawn n circles on the plane, dividing the plane into r regions. Suppose we draw another circle, forming k intersection points with the existing circles. If $k = 0$, then there are no intersection points, and the resulting number of regions is $r+1$. Otherwise, the k intersection points divide the new circle into k arcs. Each arc divides an existing region into two regions, so the resulting number of regions is $r + k$. Since the new circle intersects each other circle at most twice, we have $k \leq 2n$. By definition, $r \leq R(n)$, so the resulting number of regions is at most $R(n) + 2n$.

To show that this maximum is attainable, we need to produce a set of $n+1$ circles such that every pair of circles intersects twice, and every intersection point is distinct. It should be easy for the reader to convince himself or herself that there is such a set for all $n+1 \leq 6$ by drawing circles on paper. Therefore $R(n+1) = R(n) + 2n$ for all $n \leq 5$. We can easily see that $R(1) = 2$. Repeatedly applying the recursive equation, we obtain $R(6) = \boxed{32}$.

For a solution that doesn't require drawing circles on paper that look like they intersect properly, we can prove the following statement:

Claim: For all m, there exists a set of m circles such that every pair of circles intersects twice, and every intersection point is distinct.

Proof: Take a regular m-gon, and let s be the radius of the circumcircle of the m-gon. Put m circles at the vertices of the m-gon, all with the same radius, and let the common radius be greater than s. For two circles of the same radius to intersect twice, it is sufficient for the common radius to be greater than half the distance between the centers of the circles. Since the distance between the centers of any two of these circles is at most $2s$, the diameter of the circumcircle, and since the common radius is greater than s, every pair of circles intersects twice. Now suppose for contradiction that some three circles pass through the same point P. Then P is equidistant from three distinct points on the circumcircle of the m-gon, so P is the circumcenter of the triangle formed by these three points and thus the circumcenter of the m-gon. But then the distance between P and the three points is s. Since the common radius is greater than s, P is not on any of the circles, a contradiction. Therefore, every intersection point is distinct. Therefore, such a set of circles exists for all m.

4. Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares, and there are \$5 bills in the top right and bottom left squares, as shown below.

\$1				\$5
	E			
\$5				\$1

Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of

Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{18}$

Solution: In each square, we write the expected value of Erick's winnings starting from that square. From any square in the middle column, Erick has an equal probability of ending in the top left and top right squares, and an equal probability of ending in the bottom left and bottom right squares, by symmetry. Therefore, the total probability of Erick ending in a \$1 square is the same as the total probability of Erick ending in a \$5 square, so both probabilities are 1/2, and therefore Erick's expected winnings from any square in the middle column are \$3. By an analogous argument, Erick's expected winnings from any square in the middle row is \$3. Let x be the expected value of Erick's winnings starting from the 2nd row and 2nd column, and let y be the expected value of Erick's winnings starting from either the 1st row and 2nd column, or the 2nd row and 1st column, since these expected values are the same by symmetry.

\$1	y	\$3		\$5
y	x	\$3		
\$3	\$3	\$3	\$3	\$3
		\$3		
\$5		\$3		\$1

Then we have a system of equations:

$$\begin{aligned} x &= \frac{1}{2} \cdot y + \frac{1}{2} \cdot 3 \\ y &= \frac{1}{3} \cdot x + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 3 \end{aligned}$$

Solving yields $x = 13/5$, so the answer is $\boxed{18}$.

5. We say that a rook is "attacking" another rook on a chessboard if the two rooks are in the same row or column of the chessboard and there is no piece directly between them. Let n be the maximum number of rooks that can be placed on a 6×6 chessboard such that each rook is attacking at most one other. How many ways can n rooks be placed on a 6×6 chessboard such that each rook is attacking at most one other?

Answer: $\boxed{8100}$

Solution: Consider the following arrangement of rooks, where an R represents a rook:

R	R				
		R	R		
				R	
				R	
					R
					R

In this arrangement, each rook attacks at most one other, so $n \geq 8$. Suppose there is such an arrangement of 9 rooks. Each row has at most 2 rooks, so there must be some 3 rows with exactly 2 rooks each. Call these 6 rooks "strong."

Suppose there are two strong rooks in the same column of the chessboard. Then these rooks each attack both each other and the strong rooks they share a row with, a contradiction. Therefore, the strong rooks all occupy different columns, so there is a strong rook in each column. Since there are only 6 strong rooks, there must be some rook R on the chessboard that is not strong. Take the strong rook that is in the same column as R. This strong rook attacks both R and the strong rook it shares a row with, a contradiction. Therefore there is no such arrangement of 9 rooks, so $n = 8$.

Consider arrangements of 8 rooks. Since each row has at most 2 rooks, there are exactly 2 rows with 2 rooks and 4 rows with 1 rook. Similarly, there are exactly 2 columns with 2 rooks and 4 columns with 1 rook. The rooks in the rows with 2 rooks must have their own columns, so these are the 4 columns with 1 rook. Call these 4 rooks the "row" rooks. Similarly, the rooks in the columns with 2 rooks are the rooks in the rows with 1 rook. Call these 4 rooks the "column" rooks.

Therefore we can produce an arrangement by choosing the 2 rows that are to have 2 rooks and the 2 columns that are to have 2 rooks. The row rooks are in these 2 rows and in the other 4 columns, and the column rooks are in these 2 columns and in the other 4 rows. We finish constructing the arrangement by choosing 2 of the other 4 columns to contain the upper-most row of row rooks and 2 of the other 4 rows to contain the left-most column of column rooks.

Therefore the answer is $\binom{6}{2}^2 \binom{4}{2}^2 = \boxed{8100}$.

6. All the diagonals of a regular decagon are drawn. A regular decagon satisfies the property that if three diagonals concur, then one of the three diagonals is a diameter of the circumcircle of the decagon. How many distinct intersection points of diagonals are in the interior of the decagon?

   Answer: $\boxed{161}$

Solution: First we classify the intersection points. There are points on exactly 2 lines, points on exactly 3 lines, and the center of the decagon, which is on all 5 diameters. To prove there are no other points, suppose a point other than the center is on at least 4 lines. The point is on a diameter by the given property. Let the vertices of the decagon be $P_1, \cdots, P_{10}$ clockwise around the decagon, such that the point is on P_1P_6. Now consider all the intersection points on P_1P_6. Given a diagonal P_iP_j that intersects P_1P_6, let $f(P_iP_j)$ be the intersection point. Then the intersection points on P_1P_6 between P_1 and the center of the decagon are

$$f(P_2P_{10}), f(P_2P_9) = f(P_3P_{10}), f(P_2P_8) = f(P_4P_{10}), f(P_3P_9)$$

Of these points, $f(P_2P_{10})$ is the closest to P_1, and $f(P_2P_9) = f(P_3P_{10})$ is the next closest to P_1. The other two points are clearly further from P_1. Therefore, our point can only exist if these other two points are the same point, so that $f(P_2P_8) = f(P_4P_{10}) = f(P_3P_9)$. But by symmetry, P_2P_8 and P_3P_9 intersect on the diameter of the circumcircle of the decagon that is perpendicular to P_1P_{10} and P_5P_6. Therefore, if our point exists, then it is on two different diameters of the circumcircle of the decagon, so our point is the center of the decagon, a contradiction. Therefore our original classification of points was correct.

Now we take a PIE-like approach to count the intersection points. Each set of four points corresponds to exactly one intersection point, contributing $\binom{10}{4} = 210$ intersection points. However, we have overcounted, since some of these intersection points are actually the same point.

Consider intersection points of exactly three diagonals. By the given property, one diagonal is a diameter, and the other two diagonals are reflections of each other over the diameter, since otherwise we could reflect the diagonals over the diameter to get more diagonals that pass through the same point. We can produce such an intersection point by choosing a diameter and two of the four points on one side of the diameter, taking the corresponding two points on the other side of the diameter, and drawing the resulting diagonals. There are $5\binom{4}{2} = 30$ such diagonals. Each of the 3-line diagonals was counted $\binom{3}{2} = 3$ times, so we need to subtract 60.

At this point, we've only miscounted the center. We counted it $\binom{5}{2} = 10$ times in the first step, and we counted it $5 \cdot 2 = 10$ times in the second step, which we actually subtracted twice, so we've counted it -10 times total. Therefore, we need to add 11.

Therefore, the answer is $210 - 60 + 11 = \boxed{161}$.

7. Matt is asked to write the numbers from 1 to 10 in order, but he forgets how to count. He writes a permutation of the numbers $\{1, 2, 3 \ldots, 10\}$ across his paper such that:

   (a) The leftmost number is 1.

   (b) The rightmost number is 10.

   (c) Exactly one number (not including 1 or 10) is less than both the number to its immediate left and the number to its immediate right.

How many such permutations are there?

Answer: $\boxed{1636}$

Solution: Consider the "changes of direction" of the sequence of numbers. It must switch from decreasing to increasing exactly once by (3). By (1) and (2) it must start and end as increasing. Therefore the sequence must go from increasing to decreasing to increasing.

Let a be the unique number that's less than both its neighbors, corresponding to the switch from decreasing to increasing, and let b be the unique number that's greater than both its neighbors, corresponding to the switch from increasing to decreasing. Then the sequence is of the form $1, \ldots, b, \ldots, a, \ldots, 10$, where $1 < a < b < 10$, and the sequence is monotonic between 1 and b, between b and a, and between a and 10.

If we fix a and b, then the sequence is uniquely determined by the sets of numbers in each of the three dotted sections. In other words, we simply have to choose which of the three sections to place each of the remaining numbers. The numbers between 1 and a must go to the left of b, and the numbers between b and 10 must go to the right of a. The numbers between a and b can go in any of the three sections. For example, if $a = 2$, $b = 8$, and we divide the numbers between a and b into the sets $\{4, 6\}, \{3\}, \{5, 7\}$, then we obtain the unique permutation $1, 4, 6, 8, 3, 2, 5, 7, 9, 10$. Therefore the number of permutations is

$$N = \sum_{1<a<b<10} 3^{b-a-1}$$

For each $1 \leq n \leq 7$, if $b - a = n$, then $1 < a = b - n < 10 - n$, so there are $8 - n$ possible values of a, each of which uniquely determines a value of b. Therefore,

$$N = \sum_{n=1}^{7} (8-n)3^{n-1} = \sum_{n=0}^{6} (7-n)3^n$$

Multiplying the first expression above by 3, we obtain

$$3N = \sum_{n=1}^{7} (8-n)3^n$$

Subtracting, we obtain

$$\begin{aligned}
2N &= (8-7)3^7 - (7-0)3^0 + \sum_{n=1}^{6} 3^n \\
&= -7 + \sum_{n=1}^{7} 3^n \\
&= -7 + 3\sum_{n=0}^{6} 3^n \\
&= -7 + 3 \cdot \frac{3^7 - 1}{3 - 1} \\
&= 3272
\end{aligned}$$

and so $N = \boxed{1636}$.

8. Let N be the sum of all binomial coefficients $\binom{a}{b}$ such that a and b are nonnegative integers and $a+b$ is an even integer less than 100. Find the remainder when N is divided by 144. (Note: $\binom{a}{b} = 0$ if $a < b$, and $\binom{0}{0} = 1$.)

   Answer: $\boxed{3}$

   Solution: Group the binomial coefficients by $a+b$. Then

$$N = \sum_{n=0}^{49} \sum_{k=0}^{n} \binom{2n-k}{k}$$

The key step is to notice that the inner sum is the $2n$-th Fibonacci number F_{2n}, where F is defined by $F_0 = F_1 = 1$ and $F_{i+1} = F_i + F_{i-1}$ for all positive integers i. This can be proved by induction.

Alternatively, consider partitioning a row of $2n$ squares into segments, each consisting of 1 or 2 squares. If there are k segments of 2 squares, then there are $2n-k$ total segments, and therefore $\binom{2n-k}{k}$ such partitions. As k varies through all possible values, the inner sum above counts all such partitions. Therefore, the inner sum above is the number of partitions of a row of $2n$ squares into segments of 1 or 2 squares.

But the number of partitions of a row of m squares into segments of 1 or 2 squares is F_m. Clearly we can partition a row of 0 or 1 squares just 1 way, and the number of partitions of a row of m squares is equal to the number of partitions ending in a 1-square segment, which is the number of partitions of a row of $m-1$ squares, plus the number of partitions ending in a 2-square segment, which is the number of partitions of a row of $m-2$ squares. Therefore the numbers of partitions of rows of squares satisfies the Fibonacci recursion and has the same initial values, so the number of partitions of a row of m squares into segments of 1 or 2 squares is F_m. Therefore the inner sum is F_{2n}.

Then

$$\begin{aligned} N &= F_0 + F_2 + \cdots + F_{98} \\ &= F_1 + (F_2 + F_4 + \cdots + F_{98}) \\ &= F_3 + (F_4 + F_6 + \cdots + F_{98}) \\ &\vdots \\ &= F_{97} + (F_{98}) \\ &= F_{99} \end{aligned}$$

Now we want to evaluate $F_{99} \pmod{144}$. Note that $F_{10} = 89$ and $F_{11} = 144$. Therefore, $F_{11} \equiv 0 \pmod{144}$, $F_{12} \equiv 89 \pmod{144}$, and $F_{13} \equiv 89 \pmod{144}$. Then $F_{12} \equiv 89F_0 \pmod{144}$ and $F_{13} \equiv 89F_1 \pmod{144}$. Therefore, we find inductively that $F_{i+12} \equiv 89F_i \pmod{144}$ for all nonnegative integers i.

Therefore, since $99 = 8 \cdot 12 + 3$, we have $F_{99} \equiv 89^8 F_3 \equiv 3 \cdot 89^8 \pmod{144}$. By the Chinese Remainder Theorem, it is sufficient to calculate F_{99} modulo 16 and 9. We have

$$\begin{aligned} F_{99} &\equiv 3 \cdot 9^8 \equiv 3 \cdot 81^4 \equiv 3 \pmod{16} \\ F_{99} &\equiv 3 \cdot (-1)^8 \equiv 3 \pmod{9} \end{aligned}$$

Therefore, by the Chinese Remainder Theorem, $F_{99} \equiv 3 \pmod{144}$, so the answer is $\boxed{3}$.