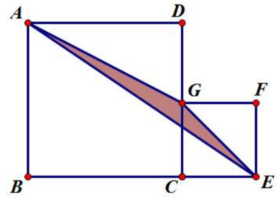
PUMaC 2010

Geometry A Solutions

1. As in the following diagram, square $ABCD$ and square $CEFG$ are placed side by side (i.e. C is between B and E and G is between C and D). If $CE = 14$, $AB > 14$, compute the minimal area of $\triangle AEG$.



[Answer] 98

[Solution] Connect AC, Note that two triangles AEG and CEG share same base and have equal height. So the area of $\triangle AEG$ is equal to area of $\triangle CEG = 14 \times 14/2 = 98$.

2. In a rectangular plot of land, a man walks in a very peculiar fashion. Labeling the corners $ABCD$, he starts at A and walks to C. Then, he walks to the midpoint of side AD, say A_1. Then, he walks to the midpoint of side CD say C_1, and then the midpoint of A_1D which is A_2. He continues in this fashion, indefinitely. The total length of his path if $AB = 5$ and $BC = 12$ is of the form $a + b\sqrt{c}$. Find $\frac{abc}{4}$.

[Answer] 793

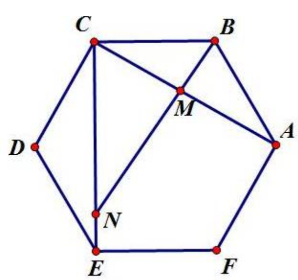
[Solution] $AC + A_1C = 13 + \sqrt{61}$. Hence total length is $(1 + 2^{-1} + 2^{-2} + \cdots)(13 + \sqrt{61}) = 26 + 2\sqrt{61}$.

3. Triangle ABC has $AB = 4$, $AC = 5$, and $BC = 6$. An angle bisector is drawn from angle A, and meets BC at M. What is the nearest integer to $100\frac{AM}{CM}$?

[Answer] 100

[Solution] By Angle-Bisector Theorem, $BM : CM = AB : AC = 4 : 5$, hence $BM = 8/3$, $CM = 10/3$. By Angle-Bisector Length Formula, $AM = \sqrt{AB \cdot AC - BM \cdot CM} = 10/3$. Hence $AM/CM = 1$.

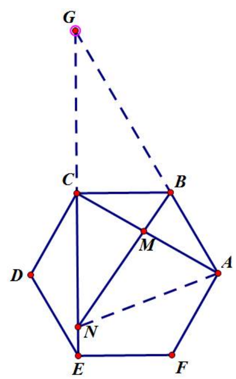
4. In regular hexagon $ABCDEF$, AC, CE are two diagonals. Points M, N are on AC, CE respectively and satisfy $AC : AM = CE : CN = r$. Suppose B, M, N are collinear, find $100r^2$.



[Answer] 300

[Solution] Let the side length of hexagon be 1. Extend NC and AB to intersect at some point G. Then $AB = 1$, $BG = 2$, $GC = \sqrt{3}$, let $CN = x$. We use $[XYZ]$ to denote area of triangle XYZ. Then

$$[BCG] : [BCN] = CG : CN = \sqrt{3} : x \qquad [BAN] : [BGN] = BA : BG = 1 : 2$$



Consequently,

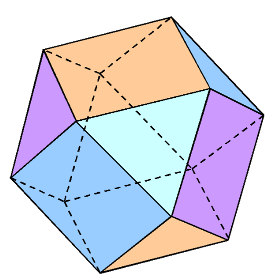
$$AM : CM = [BCN] : [BAN] = \frac{\sqrt{3}+x}{2} : x = \frac{\sqrt{3}+x}{2x}.$$

By condition, $AM : AC = CN : CE$, which translates to

$$\frac{\sqrt{3}+x}{\sqrt{3}+3x} = \frac{x}{\sqrt{3}}.$$

Solve for x, the only positive solution is 1. Hence $r = CE/CN = \sqrt{3}/1 = \sqrt{3}$.

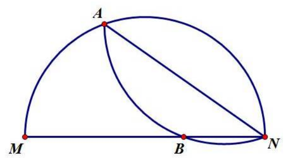
5. A cuboctahedron is a solid with 6 square faces and 8 equilateral triangle faces, with each edge adjacent to both a square and a triangle (see picture). Suppose the ratio of the volume of an octahedron to a cuboctahedron with the same side length is r. Find $100r^2$.



[Answer] 4

[**Solution**] A cuboctahedron is obtained by chopping off 8 corner tetrahedra of a cube. So volume of cuboctahedron of side length 1 is $(\sqrt{2})^3 - 8 \times \frac{1}{6}(1/\frac{2}{)}^3 = \frac{5}{3}\sqrt{2}$. On the other hand, volume of a regular octahedron is $1^2 \times \frac{1}{\sqrt{2}} \times \frac{1}{3} \times 2 = \frac{\sqrt{2}}{3}$. So the ratio is $1:5$.

6. In the following diagram, a semicircle is folded along a chord AN and intersects its diameter MN at B. Given that $MB : BN = 2 : 3$ and $MN = 10$. If $AN = x$, find x^2.



[**Answer**] 80

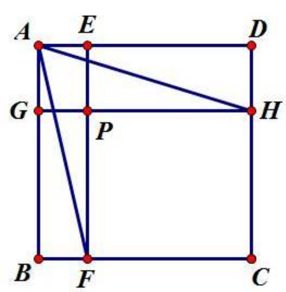
[**Solution**] Let C be symmetry point of B w.r.t. AN, then C is on arc AN with $CN = BN = 6$. Then $MC = 8$. Suppose $AN = x$, then $AM = \sqrt{10^2 - x^2}$. Also, by symmetry, $AM = AC$.

Apply Ptolemy's Theorem on cyclic quadrilateral $AMNC$ we get:

$$AM \cdot CN + MN \cdot AC = AN \cdot MC$$
$$\Rightarrow 6\sqrt{10^2 - x^2} + 10\sqrt{10^2 - x^2} = 8x$$

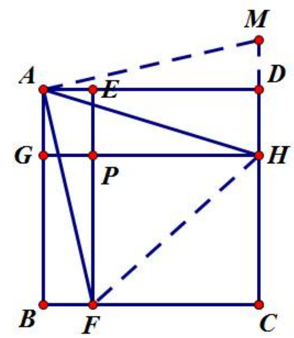
Solve for x we get $x = 4\sqrt{5}$.

7. Square $ABCD$ is divided into four rectangles by EF and GH. EF is parallel to AB and GH parallel to BC. EF and GH meet at point P. The area of rectangle $PFCH$ is twice that of rectangle $AGPE$. If maximal value of $\angle FAH$ in degrees is x, find the nearest integer to x.



[**Answer**] 45

[**Solution**] Let side length of square be 1. Let $AG = x$, $AE = y$. Then $(1-x)(1-y) = 2xy \Rightarrow x + y = 1 - xy$. There are two ways to proceed from here.



(Analytic Method) We have $\tan \angle HAD = x$, $\tan \angle FAB = y$, by compound angle formula

$$\tan(\angle HAD + \angle FAB) = \frac{x+y}{1-xy} = 1,$$

which implies that $\angle HAD + \angle FAB = 45°$, i.e. $\angle HAF = 45°$.

(Geometric Solution) Rotate $\triangle ABF$ 90 degrees counterclockwise to $\triangle ADM$. We inteprete the LHS and RHS of equation $x+y = 1-xy$ as follow: LHS is twice the sum of areas of $\triangle ABF$ and $\triangle ADH$, which is the area of $\triangle AHM$; RHS is area of pentagon $ABFHD$, which is equal to area of quadrilateral $AFHM$. Therefore, $\triangle AFH$ and $\triangle AHM$ have equal area. Also since $AF = AM$, $AH = AH$, the two triangles must be congruent, which implies $\angle FAH = 45°$.

8. There is a point light source in an empty universe. What is the minimal number of solid balls (of any size) that one must place in the universe so that any light ray emanating from the light source intersects at least one ball?

   [**Answer**] 4

   [**Solution**] It is easy to see that 3 is not enough. Since suppose we regard the plane passing through light source and parallel to the triangle formed by centers of three balls as equatorial plane, then all high-lattitude light rays cannot be blocked.

   To show 4 is enough, consider a regular tetrahedron ABCD with light source O at center. There exist four infinite cones containing tetrahedra OABC, OABD, OACD, OBCD. Any inscribed ball in a cone will block all light rays in that cone. Choose four non-intersecting balls inscribed in the four cones will give us the solution. The non-intersecting condition can be met since the four cones are infinite, and we can stipulate the radii of 4 balls to be drastically different. To be precise, for an inscribed ball with radius R, there exists α and β such that the ball lies entirely in the interior of the spherical shell centered at origin with inner radius αR and outer radius βR. Then we choose the fall balls with radius $R, \alpha\beta^{-1}R, \alpha^2\beta^{-2}R, \alpha^3\beta^{-3}R$. This way, the four balls will not intersect since they lie in four different spherical shell regions.