



Number Theory A Solutions

- Find the smallest positive integer n such that $n^4 + (n + 1)^4$ is composite.
Solution: One can check that 17, 97, 337, and 881 are prime, and also that $17 \cdot 113 = 1921$, so the answer is 5.
- Find the largest positive integer n such that $\sigma(n) = 28$, where $\sigma(n)$ is the sum of the divisors of n , including n .
Solution: The answer is 12. It isn't hard to see that that is, in fact, the only value where that equation holds.
- Find the sum of the first 5 positive integers n such that $n^2 - 1$ is the product of 3 distinct primes.
Solution: The five numbers are 14, 16, 20, 22, and 32, for an answer of 104.
- Find the largest positive integer n such that $n\varphi(n)$ is a perfect square. ($\varphi(n)$ is the number of integers k , $1 \leq k \leq n$ that are relatively prime to n)
Solution: If $n > 1$, then the highest prime that divides n can be shown to divide $n\varphi(n)$ to an odd power, and so $n\varphi(n)$ cannot be a perfect square. It is easy to see that 1 is a perfect square.
- Given that x, y are positive integers with x as small as possible, and y minimized with that constraint, and $x(x+1)|y(y+1)$, but neither x nor $x+1$ divides either y or $y+1$, find $x^2 + y^2$.
Solution: $x = 14, y = 20$. These are the first two times that n and $n+1$ are not powers of primes. The answer is then easily seen to be 596.
- Find the numerator of

$$\frac{1010 \overbrace{11 \dots 11}^{2011 \text{ ones}} 0101}{1100 \underbrace{11 \dots 11}_{2011 \text{ ones}} 0011}$$

when reduced to lowest terms.

Solution: Note that $1 + x^2 + x^4 + x^5 + \dots + x^{2n+1} + x^{2n+2} + x^{2n+4} + x^{2n+6} = (1 - x + x^2 - x^3 + x^4)(1 + x + \dots + x^{2n+1} + x^{2n+2})$, as well as $1 + x + x^4 + x^5 + \dots + x^{2n+1} + x^{2n+2} + x^{2n+5} + x^{2n+6} = (1 - x^2 + x^4)(1 + x + \dots + x^{2n+1} + x^{2n+2})$. The easiest way to see this is to either multiply numerator and denominator by $x - 1$, or to just numerically plug in small odd values of 2011 in the original equation. Plugging in $n = 1006$ and $x = 10$ gives the fraction as $\frac{9091}{9901}$.

- Let $I = \{0, 1, 2, \dots, 2008, 2009\}$, and let $S = \{f : I \rightarrow I \mid f(a) \equiv g(a) \pmod{2010} \forall a\}$, where $g(a)$ ranges over all polynomials with integer coefficients. The number of elements in S can be written as $p_1 p_2 \dots p_k$, where the p_i s are (not necessarily distinct) primes. Find $p_1 + p_2 + \dots + p_n$.
Solution: Any polynomial is determined, by the Chinese remainder theorem, by its restriction to the integers mod 2, 3, 5, and 67. There are thus at most $2^2 \cdot 3^3 \cdot 5^5 \cdot 67^{67}$ such functions. However, if you have a polynomial in those various moduli, then there is a polynomial that restricts to those polynomials, so there are at least $2^2 \cdot 3^3 \cdot 5^5 \cdot 67^{67}$ such functions and thus exactly $2^2 \cdot 3^3 \cdot 5^5 \cdot 67^{67}$ such functions. The answer is then 4527.



8. An awesome pythagorean triple is a pythagorean triple of the form $a^2 + (a + 1)^2 = b^2$, a and b positive integers. Given that a , $a + 1$, and b form the third positive awesome pythagorean triple, find a .

Solution: The equation is $2a^2 + 2a + 1 = b^2$, which can be rearranged to $(2a + 1)^2 - 2b^2 = -1$. The solutions to that are given by the coefficients of $(1 + \sqrt{2})(3 + 2\sqrt{2})^n$, and one gets a values 0 (trivial and ignored), 3, 20, and 119, so 119 is the answer.