PUMaC 2010 Power Round:
Graph Minors—Answers

These rules supersede any rules appearing elsewhere about the Power Test.

For each problem, you may use without proof the results of all previous problems (that is, problems that appear earlier in the test), even if your team has not solved these problems. You may cite results from conjectures or subsequent problems only if your team solved them independently of the problem in which you wish to cite them. You may not cite parts of your proof of other problems: if you wish to use a lemma in multiple problems, reproduce it in each one.

It is not necessary to do the problems in order, although it is a good idea to read all the problems, so that you know what is permissible to assume when doing each problem. However, please collate the solutions in order in your solution packet. Each section should start on a new page.

Using printed and noninteractive online references, computer programs, calculators, and Mathematica (or similar programs), is allowed. (If you find something online that you think trivializes part of the problem that wasn't already trivial, let us know—you won't lose points for it.) No communication with humans outside your team about the content of these problems is allowed.

Each problem is marked with a number of stars. This is both the test-writers' estimate of its relative difficulty and an indication of what sort of answer we expect:

$\star$ problems need a short (probably one-sentence) explanation. The explanation should be just detailed enough that we couldn't explain anything incorrect by it.

$\star\star$ problems need a proof. Partial credit will be given for ideas useful in a correct proof.

$\star\star\star$ problems need a proof, and we reserve half their credit for the elegance of the proof.

$\star\star\star\star$ problems have the same rules as $\star\star$ problems.

$\star\star\star\star\star$ problems need an elegant proof. No points are awarded for an inelegant one.

Note that giving an answer less rigorous than we expect is worth 0 points: for instance, if a problem asks you to find a graph with certain properties, giving the graph alone is worth nothing unless you also prove that it has those properties.

Each problem also has a maximum possible score listed after its star rating. The total number of points available is 1,000,100. Power test scores will be scaled before use in the rest of PUMaC.

1 Basic definitions

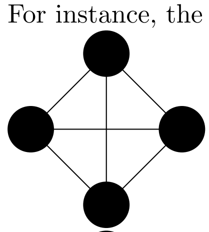
Definition 1.1. A multiset is a set in which repeated elements are allowed. The order of the elements does not matter, but the multiplicities do.

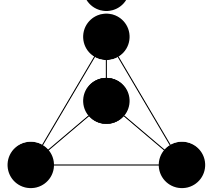
Definition 1.2. A graph G is a multiset $E(G)$ of submultisets ("edges") of size 2 of a set $V(G)$ ("vertices").

For instance, $\{\{1,1\},\{1,1\},\{1,3\},\{1,3\},\{2,3\},\{4,5\}\}$ is a simple graph on 5 vertices 1, 2, 3, 4, and 5.

For ease of use, we often draw pictures of graphs with the vertices as dots and the edges as line segments connecting them, and say that two vertices are adjacent if they're both elements of some edge.

For instance, the following are two pictures of K_4:



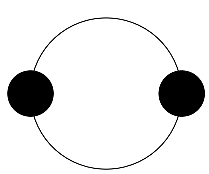


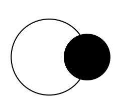
You can use these pictures to represent graphs wherever they occur, but keep in mind that a graph is defined by sets of vertices and edges, not by its drawings.

Two equal edges of a graph are called parallel edges, and an edge with two equal vertices is called a loop.

Definition 1.3. The simplification of a graph G is the graph H with all loops removed, and all but one of each set of parallel edges removed. A graph is simple if it has no loops or parallel edges.

Here are drawings of the most basic graphs that are not simple:





A few special types of graphs deserve mention:

- A complete graph K_n is the simple graph on n vertices, every pair of which are adjacent.

- A complete bipartite graph on m and n vertices $K_{m,n}$ is the simple graph with $m+n$ vertices, with each of the first m vertices adjacent to each of the last n (for a total of mn edges).
- A cycle C_n has n vertices $v_1, \ldots, v_n$, an edge $\{v_i, v_{i+1}\}$ for each $1 \leq i < n$, and an edge $\{v_1, v_n\}$. So for all n, C_n has n edges.
- A wheel W_n is C_n plus an extra vertex adjacent to all the vertices of C_n (by one edge).

Definition 1.4. A graph G is disconnected if and only if its vertices can be divided into two nonempty sets A and B such that no vertex of A is adjacent to any vertex of B. Otherwise, G is connected.

Definition 1.5. A path between two vertices u and v is a sequence of distinct vertices $v_0 = u, v_1, \ldots, v_k = v$ such that for every $0 \leq i < k$, v_i is adjacent to v_{i+1}. The length of such a path is k, the number of edges.

Problem 1.1. ($\star$, 2) For which values of n is C_n simple?

Answer: $n \geq 3$. If $n = 1$, it's a (nonsimple) loop; if $n = 2$, it's a (nonsimple) pair of parallel edges; if $n \geq 3$, it's simple.

Problem 1.2. ($\star$, 1) How many edges does W_6 have?

Answer: 12.

Problem 1.3. ($\star\star$, 4) Prove that a graph G is connected if and only if there's a path between every pair of distinct vertices.

Answer: We'll prove that G is disconnected if and only if there exist distinct vertices u and v with no path between them. If there exist such vertices, let A be the set of vertices u' such that there's a path from u to u', and let B be the remaining vertices. Then $u \in A$ and $v \notin A$ by assumption, so both A and B are nonempty, and there are no edges between A and B because if there were an edge between $u' \in A$ and $v' \in B$, then, since there's a path between u and u', there'd be a path between u and v', and v' would be in A, contradiction. Conversely, if G is disconnected, let (A, B) be a partition of its vertices with no edges between A and B, and choose $u \in A$ and $v \in B$. Then if there's a path between them, its first vertex is in A and its last is in B, so some pair of adjacent vertices has one in A and the other in B, giving an edge between A and B, contradiction. Hence G is connected if and only if there's a path between every pair of distinct vertices, as desired.

Definition 1.6. If G is a graph and A is a subset of its vertices, then the subgraph of G induced on A, denoted $G|_A$, is the graph whose vertex set is A such that for any $u, v \in A$, the multiplicity of the edge $\{u, v\}$ in $E(G|_A)$ is the same as the multiplicity of $\{u, v\}$ in $E(G)$.

Definition 1.7. If G is a graph and $v \in V(G)$, then $G \setminus v$ ("G delete v") is $G|_{V(G)\setminus\{v\}}$.

Problem 1.4. ($\star$, 1) What graph do you get by deleting a vertex of K_n?

Answer: K_{n-1}.

Problem 1.5. ($\star$, 4) How many distinct graphs can you get by deleting three vertices of W_{2010}? Note that, say, two copies of K_3 aren't distinct, even if they came from different vertices of a bigger graph.

Answer: $\frac{2010}{2} + \frac{\binom{2010}{3}+2*\frac{2010}{3}+\frac{2010}{2}*2*(\frac{2010}{2}-1)}{2*2010} = 337725$: there are $\frac{2010}{2}$ distinct subgraphs with the center deleted; for the rest, apply the orbit-stabilizer theorem, in which the identity fixes everything, two rotations by 667 vertices fix 667 colorings, and 1005 reflections fix $2 * 1004$ each.

Definition 1.8. If G is a graph and $e \in E(G)$, then $G \setminus e$ ("G delete e") is the graph with the same vertices as G, and the same multiset of edges except we remove one copy of e.

Definition 1.9. A graph H is a subgraph of a graph G if you can get from G to H by deleting vertices (i.e., taking an induced subgraph) and then deleting edges.

Problem 1.6. ($\star$, 1) Prove that if a graph G with $n \geq 0$ vertices doesn't have K_1 as a subgraph, then it has at most $-n$ edges.

Answer: If G doesn't have K_1 as a subgraph, then it has no vertices ($n = 0$), and hence no edges, that is, at most $-n = 0$.

2 Edge contraction

Definition 2.1. If G is a graph and $e = \{u, v\} \in V(G)$ (with u and v not necessarily distinct), then G/e ("G contract e") is the subgraph of G induced on all its vertices but u and v, plus an extra vertex ϵ, and some new edges containing ϵ: each edge of G of the form $\{u, a\}$ or $\{v, a\}$, for $a \notin \{u, v\}$ is replaced with an edge $\{\epsilon, a\}$. Each edge of G of the form $\{u, v\}$, $\{u, u\}$, or $\{v, v\}$, is replaced with a loop at ϵ, except for e itself.

To picture this, you can imagine that the vertices are big blobs of clay and each edge is a thin cable connecting two blobs. When you contract an edge, you mold the cable and the two blobs it connected into one new blob, without destroying the other cables. For example, if you contract an edge of K_5, you get K_3 plus one other vertex sharing two edges with each vertex in K_3. If you contract an edge of $K_{3,3}$ you get W_4.

Problem 2.1. ($\star$, 2)

- True or false: You can contract an edge of W_4 to get W_3.
- True or false: If an edge e is a loop (that is, it's $\{v, v\}$ for some vertex v), then $G/e = G \setminus e$.

Answer:

1. False: W_4 has 8 edges, W_3 has 6 edges, and contracting an edge reduces the number of edges by exactly 1

2. True: Check the definition.

Definition 2.2. A graph H is a minor of a graph G if and only if you can "reach H from G by repeatedly deleting a vertex, deleting an edge, or contracting an edge." That is, if and only if there's a sequence of graphs $G_0 = H, G_1, G_2, \ldots, G_{k-1}, G_k = G$ such that for each $0 \le i < k$, either $G_i = G_{i+1} \setminus v$ for some vertex v, or $G_i = G_{i+1} \setminus e$ for some edge e, or $G_i = G_{i+1}/e$ for some edge e.

Problem 2.2. ($\star\star$, 6) Prove that H is a minor of G if and only if you can map each vertex $v \in V(H)$ to a nonempty subset of G's vertices $f(v) \subset V(G)$ such that:

- For all $v \in V(H)$, the subgraph of G induced on $f(v)$ is connected.

- For all $u, v \in V(H)$ with $u \neq v$, $f(u) \cap f(v) = \{\}$.

- For all $u, v \in V(H)$, $|E(f(u), f(v))| - |f(u) \cap f(v)| \ge E(\{u\}, \{v\}) - |\{u\} \cap \{v\}|$, where $E(X, Y)$ is the number of edges with one endpoint in X and the other in Y.

Answer (sketch): First, we show that if H is a minor of G, the above holds, by induction on k in the definition of minors. If $k = 0$, $H = G$, and we can take $f(v) = v$ to satisfy the above. If not, then either $G \setminus v = G_k$, or $G \setminus e = G_k$, or $G/e = G_k$, for some G_k satisfying the above and some v or e. In the first two cases, the same f as the inductive hypothesis gives for G_k works; in the third case, let f map both vertices of e to $f(e)$.

Second, we show that if the above holds, then H is a minor of G. For each $v \in V(H)$, contract all the edges in the subgraph of G induced on $f(v)$; this produces a single vertex since $f(v)$ is connected, and for $u \neq v$, these vertices are distinct, since $f(v) \cap f(u) = \{\}$. Delete any unused vertices and delete any edges between $f(u)$ and $f(v)$ in excess of the number of edges between u and v in H; this gives H as a minor of G, as desired.

Problem 2.3. ($\star$, 3)

- Is K_5 a minor of K_8?

- Is K_5 a minor of $K_{2,2}$?

- Is K_5 a minor of $K_{3,3}$?

- Is K_5 a minor of $K_{4,4}$?

- Is K_5 a minor of $K_{5,5}$?

- Is K_5 a minor of C_{125}?
- Is K_4 a minor of W_3?
- Is K_4 a minor of W_5?
- Is K_5 a minor of W_6?

Answer:

1. Yes. Delete any three vertices.
2. No. $K_{2,2}$ has fewer vertices.
3. No. $K_{3,3}$ has fewer edges.
4. Yes. Pick three nonintersecting edges and contract them.
5. Yes. Pick five nonintersecting edges and contract them.
6. No. Every minor of C_{125} has no vertices of degree more than 2.
7. Yes. They're equal.
8. Yes.
9. Yes.

Problem 2.4. ($\star$, 2 points)

- What is the smallest n such that K_{2010} is a minor of $K_{n,n}$?
- What is the smallest n such that W_{2010} is a minor of $K_{n,n}$?

Answer:

1. 2009. At most one singleton vertex from each partite set of $K_{n,n}$ can be mapped to a vertex of K_{2010}, so $K_{n,n}$ must have at least $2 * (2010 - 2) + 2$ vertices. 2009 suffices: contracting 2008 nonintersecting edges of $K_{2009,2009}$ gives K_{2010}.
2. 1006. It's at least 1006 because $K_{1005,1005}$ has fewer vertices than W_{2010}, and $K_{1005,1005}$ suffices by deleting any vertex, contracting any remaining edge, and then deleting some edges.

Problem 2.5. ($\star$, 1) Prove that if a simple graph G with $n \geq 1$ vertices has no K_2 minor, then it has at most 0 edges.

Answer: "No K_2 minor" means no edges.

3 Graph colorings

Definition 3.1. A graph is k-colorable if and only if you can partition its vertices into k (possibly empty) subsets such that no vertices within a subset are adjacent. (Intuitively, the subsets are colors, and no two vertices of the same color are adjacent.)

Problem 3.1. ($\star$, 1) Find a graph that's not 3-colorable.

Answer: For instance, K_4.

Hadwiger's conjecture states that if G is a simple graph with no K_t minor, then G is $(t-1)$-colorable.

Problem 3.2. ($\star\star$, 4) Prove Hadwiger's conjecture for $t = 3$.

Answer: A graph with no K_3 minor can't contain a cycle of length at least 3. A simple graph can't contain cycles with lengths 1 or 2, by a previous question. Hence any simple graph with no K_3 minor is a forest, which is 2-colorable.

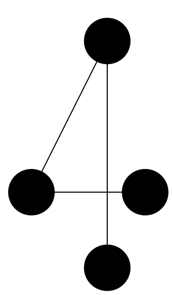
Problem 3.3. ($\star$, 2) For each $n \geq 2$, find a graph G with n vertices, more than $n-1$ edges, and no K_3 minor.

Answer: For instance, n disjoint loops.

4 Planar graphs

Definition 4.1. A graph is planar if and only if it can be drawn in the plane with no two edges intersecting. (Edges needn't be straight lines, although it happens that every simple planar graph can be drawn with straight-line edges and no edges intersecting.)

Problem 4.1. ($\star$, 3) Which of the following graphs are planar?

- K_4
- C_4
- W_4
- $K_{4,4}$
- 

Answer:

1. Yes. It had a planar drawing above.
2. Yes.
3. Yes.
4. No. In fact, even $K_{3,3}$ isn't planar, as we'll prove below.
5. Yes. A graph is defined by vertices and edges, not one drawing; this graph can be drawn planarly.

Problem 4.2. ($\star\star\star\star\star$, 1,000,000) Prove that every loopless planar graph is 4-colorable. (This is called the Four-color theorem).

Answer: (Removed to save ~~paper~~ bytes.)

Problem 4.3. ($\star\star$, 4) Prove that if a simple graph G with $n \geq 3$ vertices has no K_4 minor, then it has at most $2n - 3$ edges.

Answer: Suppose not, so there's at least one such graph G. Let G be such a graph with $|V(G)| + |E(G)|$ (the sum of the number of vertices of G, $|V(G)|$, and the number of edges of G, $|E(G)|$) minimal. If G has only 3 vertices, then it has at most $3 \leq 2(3) - 3$ edges. If G (with $|V(G)| > 3$) has any vertices of degree 0, 1, or 2, deleting them gives a graph with $n - 1 \geq 3$ vertices and more than $(2n - 3) - 2 = 2(n - 1) - 3$ edges, contradicting the minimality of G. If G has a vertex v of degree at most 3, not all of its neighbors are adjacent, or we'd have a K_4 subgraph. So, one of its neighbors, u, is adjacent to at most 1 other. Delete that edge, if it exists, and contract the edge $\{u, v\}$; that gives another simple graph with $n-1$ vertices and more than $(2n-3)-2$ edges, contradicting the minimality of G. So every vertex of G has degree at least 4. But then G has at least $2n$ edges, and we can delete any of them to get a get a smaller simple graph with no K_4 minor and at least $2n - 3$ edges. Hence no such G exists, as desired.

5 Excluded Minor Theorems

Problem 5.1. ($\star\star$, 4) Prove that K_5 and $K_{3,3}$ are not planar.

Answer: Suppose for contradiction that K_5 were planar. It has 5 vertices and 10 edges, so by Euler's formula it has 7 faces. Each face has at least 3 edges, and each edge is in at most 2 faces, for a total of at least 10.5 edges, contradiction.

Suppose for contradiction that $K_{3,3}$ were planar. It has 6 vertices and 9 edges, so by Euler's formula it has 5 faces. Each face has at least 4 edges (since $K_{3,3}$ has no triangles), and each edge is in at most 2 faces, for a total of at least 10 edges, contradiction.

Problem 5.2. ($\star\star$, 2) Prove that if G has a K_5 or $K_{3,3}$ minor, then G isn't planar.

Answer: If G is planar, $v \in V(G)$, and $(u, v) \in E(G)$, then $G \setminus v$ is planar (with the same planar drawing), $G \setminus (u, v)$ is planar (with the same planar drawing), and $G/(u, v)$ is planar: from a planar drawing of G, reroute all the edges with an endpoint u except (one of) the edge(s) (u, v) through that edge (u, v). So if G is planar, every minor of G is too, so if G has a K_5 or $K_{3,3}$ minor, it's not planar, as desired.

A famous theorem of Wagner's (sometimes attributed to Kuratowski, although he actually proved something else) states that a graph is planar if and only if it has no K_5 minor and no $K_{3,3}$ minor. The other direction of the proof is easily googleable, if you're curious.

Many classic types of graphs can be described easily in terms of minors that they don't have. For instance:

- Graphs that don't have C_1 (that is, the one-vertex, one-edge graph) as a minor are forests.
- Graphs that don't have C_2 as a minor are forests with loops allowed.
- Graphs that don't have C_3 as a minor are forests with loops and parallel edges allowed.
- Graphs that don't have a path of length 2 as a minor are matchings (plus isolated vertices, loops, and parallel edges).

Problem 5.3. ($\star\star$, 4) What graphs don't have a path of length 3 as a minor?

Definition 5.1. A k-sum of two simple graphs G and H takes a K_k subgraph of each one, identifies the vertices in them (that is, glues the K_ks together), and deletes the edges of the K_k. For instance, the only graph obtainable as a 3-sum of two K_4s is $K_{2,3}$.

Problem 5.4. ($\star$, 2) How many distinct graphs can you get as 2-sum of $K_{2,3}$ and W_4?

Answer: 3. All the edges of $K_{2,3}$ are the same, but with two possible orientations. There are two distinct edges of W_4: "spokes," which have orientation, and "treads," which don't. Hence there are at most three (identify with a tread, or with a spoke in one of the two possible orientations), and by drawing these one sees that all three are distinct.

Problem 5.5. ($\star\star$, 4) Prove that if simple G and H don't have $K_{3,3}$ as a minor, then no 0-sum, 1-sum, or 2-sum of G and H has $K_{3,3}$ as a minor.

Answer: Suppose for contradiction that a 0-sum, 1-sum, or 2-sum of graphs G and H with no $K_{3,3}$ minor has a $K_{3,3}$ minor. Let C (of size at most 2) be the k-vertex subset of the sum where the K_ks were glued together, and let G' and H' be the remainders of G and H in the sum.

We claim that at least one of the sets $f(v)$ from the definition in Problem 2.2 is contained in H'. Suppose not; then we'll show that G has a $K_{3,3}$ minor. For any $v \in K_{3,3}$ for which the set $f(v) \in G + H$ doesn't intersect H', choose the same set $f(v)$ in G; for any v for which $f(v)$ intersects H' (but isn't contained in it, by assumption), it intersects C; restrict that set to $G' \cup C$. Then all the conditions for G to have a $K_{3,3}$ minor in Problem 2.2 are satisfied except possibly adjacency between the vertices v such that $f(v)$ intersected H', but for every such v, $f(v)$ contains a vertex in C, and all the vertices of C are adjacent in G. So G has a $K_{3,3}$ minor, contradicting the assumption that none of the sets $f(v)$ from the definition of Problem 2.2 is contained in H'.

Similarly, at least one of the sets $f(v)$ from the definition in Problem 2.2 is contained in G'

Delete the (at most two) vertices in C, and the (at most two) vertices v of $K_{3,3}$ such that $f(v)$ contained a vertex in C. This disconnects $G + H$, and by the first part of this proof there's a set $f(v)$ on each side of the disconnect, so it disconnects $K_{3,3}$. But there are no two vertices of $K_{3,3}$ such that deleting them disconnects $K_{3,3}$, contradiction. Hence $G + H$ has no $K_{3,3}$ minor, as desired.

Problem 5.6. ($\star\star$, 2) Is the same true for 3-sums?

Answer: No. Neither K_5 nor K_4 has $K_{3,3}$ as a minor, but the unique graph that's a 3-sum of K_5 and K_4 is $K_{3,3}$ plus three edges.

Problem 5.7. ($\star\star\star$, 8) Prove that if a simple graph G with $n \geq 4$ vertices has no K_5 minor, then it has at most $3n - 6$ edges.

Answer: As in the previous section's final problem, let G be a counterexample with $|V(G)| + |E(G)|$ minimal. If G has only 4 vertices, then it has at most $6 \leq 3(4) - 6$ edges. If it has any vertices of degree 0, 1, 2, or 3, they can be deleted. If it has a vertex v of degree 4, not all its neighbors are adjacent, so one of them, u, is adjacent to at most 2 of v's neighbors; by deleting those at most 2 edges and contracting $\{u, v\}$, we get a smaller counterexample. If every vertex has degree at least 6, then there are at least $3n$ edges, and we can delete one to get a smaller counterexample. So it has a has a vertex v of degree 5. Let u_1, u_2, u_3, u_4, and u_5 be its neighbors. If one of the us is adjacent to only two others, we can delete and contract as above. If not, we claim that G has a K_5 minor: every u is adjacent to at least 3 others, and is nonadjacent to at most 1. Hence at most two pairs of us are nonadjacent, say $\{u_1, u_2\}$ and $\{u_3, u_4\}$. But then contracting the edge $\{u_1, u_3\}$ gives a K_5 minor, contradiction.

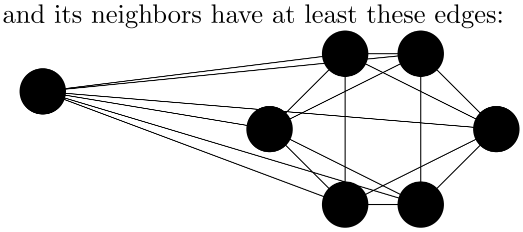
6 Reminder

Problem 6.1. (, 1) On every page you submit, put your team's name, a page count, and solutions to problems from at most one section.

Answer = Problem.

Problem 6.2. ($\star\star\star\star$, 16) Prove that if a simple graph G with $n \geq 5$ vertices has no K_6 minor, then it has at most $4n-10$ edges.

Answer: As in the previous section's final problem, let G be a counterexample with $|V(G)|+|E(G)|$ minimal. If G has only 5 vertices, then it has at most $10 \leq 4(5)-10$ edges. If it has any vertices of degree 0, 1, 2, 3, or 4, they can be deleted. If it has a vertex v of degree 5, not all its neighbors are adjacent, so one of them, u, is adjacent to at most 3 of v's neighbors; by deleting those at most 3 edges and contracting $\{u,v\}$, we get a smaller counterexample. If there's a vertex of degree 6 one of whose neighbors is adjacent to at most 3 others, then we can delete and contract similarly. If there's a vertex of degree 6 all of whose neighbors (u_1, u_2, u_3, u_4, u_5, and u_6) are adjacent to at least 4 others, that is, v and its neighbors have at least these edges:



If there's a path between any of the three pairs of vertices not shown to be adjacent here (say $\{u_1,u_4\}$, $\{u_2,u_5\}$, and $\{u_3,u_6\}$, then contracting it down to an edge and contracting one more edge gives a K_6 minor, as in the previous problem.

The new idea we introduce for this solution is a complete cutset of the graph: a subset of vertices that's complete (each vertex is adjacent to each other one) and a cutset (its removal disconnects the graph). For instance, in the case in progress (with a vertex of size 6), three mutually adjacent neighbors of v (say, the rightmost three in the picture, or $\{u_4,u_5,u_6\}$) are a clique cutset of size 3: they're all adjacent to each other, and removing them disconnects the graph since there's no path between some other neighbor w of u_4 (which exists because u_4 has degree at least 6) and u_1.

If there's a clique cutset $C \subset V(G)$ of size 3, and A and $V(G) \setminus (A \cup C)$ are nonempty subsets of $V(G)$ with no edges between them, then $G|_{A\cup C}$ and $G|_{V(G)\setminus A}$ are each smaller graphs with no K_6 minor, so they have at most $4(|A|+|C|)-10$ and $4(|V(G)|-|A|)-10$ edges, respectively. Hence the total number of edges in the graph is at most $4(|C|)-10+4(|V(G)|)-10 = 4|V(G)|-8$, but this double-counts the three edges in C, for a total of at most $4n-11$ edges in G. So, G has no vertex of degree 6.

So, every vertex of G has degree at least 7. Also, if every vertex of G has degree at least 8, then there'd be at least $4n$ edges, and we could delete some to get a smaller counterexample. So G has some vertex v of degree 7. Each of v's neighbors is adjacent to at least 4 others, or we could contract to get a smaller counterexample as above. We claim that v's neighbors contain a K_5 minor (so G contains a K_6). Indeed, the missing edges are either a subset of C_7, a subset of C_6, a subset of C_5 plus a disjoint edge, a subset of a disjoint union of a C_4 and a C_3, or a subset of C_3 plus two disjoint edges, but in all these cases there's a K_5 minor of the neighbors, and hence a K_6 minor of G, contradiction.

7 The End.

Problem 7.1. ($\star\star$, 4) For each $n \geq 6$, construct a simple graph G with n vertices, at least $5n - 15$ edges, and no K_7 minor.

Answer: Take a K_5, and $n - 5$ vertices each adjacent only to the vertices of that K_5. This has $5n - 15$ edges and is simple. If it had a K_7 minor, then at most 5 of the sets $f(v)$ from the definition in Problem 2.2 would contain vertices from the original K_5, so at least 2 of them wouldn't, but no two such sets are adjacent.

8 Credits

Problem 8.1. ($\star\star\star$, 12) For some $n \geq 7$, find a simple graph G with n vertices, more than $6n - 21$ edges, and no K_8 minor.

Answer: Start with K_{10} and delete 5 disjoint edges. This has $40 > 6(10) - 21$ edges and is simple. If it had a K_8 minor, then at most 2 of the sets $f(v)$ contain two vertices, so at least 6 of them contain only 1 vertex, so there'd be 6 vertices all adjacent to each other. But if we delete only 4 vertices of the graph, we delete endpoints of at most 4 of the removed edges, so both endpoints of one nonedge remain, contradiction. Hence there's no K_8 minor.

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