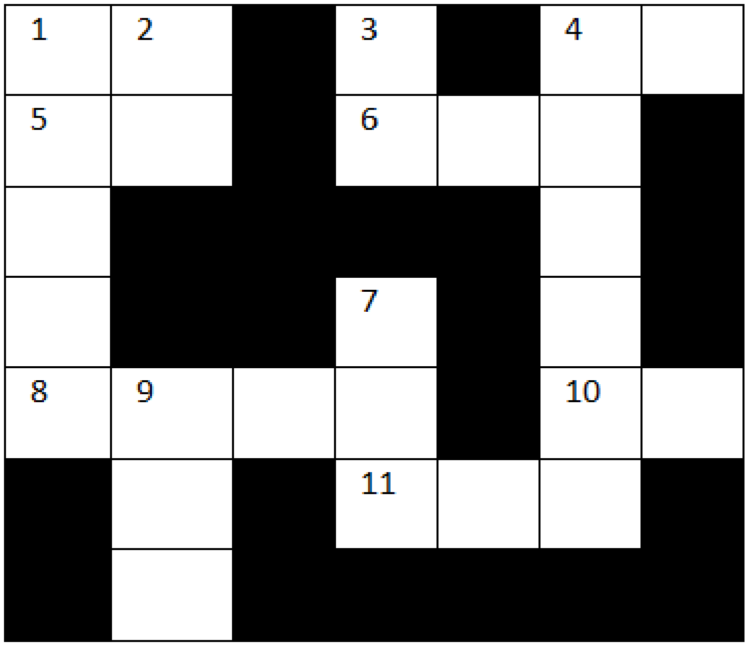
PUMaC 2010

The Team Test

The crossword is provided here as a convenience. Only your team's Master Copy will be scored.



1 Across

- **1 Across** Tetrahedron $ABCD$ has base $\triangle BCD$ which is an equilateral triangle with side length 2. $AB = AC = 3$ and $AD = 4$. Let M be the centroid of tetrahedron $ABCD$ and N be the centroid of $\triangle BCD$. Write $\dfrac{AN}{AM} = \dfrac{p}{q}$, where p and q are relatively prime integers. You are to submit the concatenation of p and q.
- **4 Across** You have 2010 ordinary decks of cards. You shuffle them all together. You then draw the least number of cards from this gigantic combined pack of 2010×52 cards so that you are guaranteed to have a four-of-a-kind (i.e. four cards of the same rank. They do *not* have to be from different suits in this scenario). Suppose you should draw k cards for some integer k. What are the last two digits of k?
- **5 Across** f is a quadratic polynomial such that $f(1) = -31$, $f(2) = -33$, and $f(5) = -27$. Find $f(10)$.

- **6 Across** Macky found two angles a and b, and the two relations

$$\cos a + \cos b = 5\sqrt{\frac{2}{29}} \qquad \sin a + \sin b = \frac{4}{\sqrt{29}}.$$

Given these two pieces of information, Macky wants to find the value of $\cos(a-b)$, which to his surprise turns out to be a positive rational number, i.e. a positive fraction of the form m/n, for integers m and n with $n \neq 0$. He reduces this fraction to its lowest form and concatenates the numerator and the denominator together. What is the final result he obtains?

- **8 Across** Ashwath decides to roll two red dice and two blue dice, as well as flip a coin. Let R be the sum of the values of the two red dice, and B be the values of the two blue dice. Tengyao will give Ashwath \$100 if $49|RB$ and if both coins come up with the same side. If the odds that Tengyao keeps his money is $\frac{p}{q}$, submit the concatenation of p and q.

- **10 Across** What is the maximum number of angles greater than π that a 45-gon can have?

- **11 Across** An urn contains a positive number of colored balls. You have no idea how many colors there are, but you do know that there are an equal number of balls of each color. You also know that adding 20 balls of a new color to the urn would not change the probability of drawing (without replacement) two balls of the same color. Before these extra balls are added, you decide to steal one ball from the urn. At *this* point, i.e. after your masterful theft of a ball and before 20 new ones are put in, how many balls are in the urn?

2 Down

- **1 Down** What 5-digit number has the property that if we put the numeral 1 at the beginning of the number, we get a number that is three times smaller than what we get if we put the numeral 1 at the end of the number?

- **2 Down** You are given five *distinct* integers a, b, c, d and e. You find that you have the equation

$$(8-a)(8-b)(8-c)(8-d)(8-e) = 28.$$

What is the value of $a+b+c+d+e$?

- **3 Down** Suppose Alex, Bob, Charles, David, Evan, Frankenstein and Gary are seven friends who want to watch a movie. However, at the theater, only one row has seven adjacent empty seats $P1$ through $P7$ left, with $P1$ and $P7$ both aisle seats (and none of the others are aisle seats). They find out a few facts about themselves, which are given as follows. Bob wants precisely one (filled) seat between himself and Charles. Gary and Charles want aisle seats, but Gary won't sit in $P1$. With these constraints, how many ways can the seven friends sit in the theater?

- **4 Down** Arthur tells you to guess the 6-digit natural number he is thinking of. The only hint he gives is that the sum of all the digits is 43. Further persuasion leads him to reveal that it is a perfect square and less than 500000.

  What is Arthur's number?

- **7 Down** Right triangle $\triangle ABC$ has integer side lengths, one side of length 29, and maximal perimeter. What is the length of the hypotenuse of this triangle?

- **9 Down** Each of 100 Oxford dons has an item of gossip known only to himself. Whenever a don telephones another, they exchange all items of gossip they know. What is the minimum number of calls they have to make in order to ensure that every one of them knows all the gossip there is to know?