1. Let the operation ★ be defined by $x ★ y = y^x - x * y$. Calculate $(3 ★ 4) - (4 ★ 3)$.

Solution: -17. We have $(3 ★ 4) - (4 ★ 3) = (64 - 12) - (81 - 12) = -17$.

2. Let $p(x) = x^2 + x + 1$. Find the fourth smallest prime $q$ such that $p(x)$ has a root mod $q$.

Solution: 19. One can check that there are roots mod 3, 7, 13, and 19, and no others for smaller primes.

3. Write $\frac{1}{\sqrt[5]{2} - 1} = a + b\sqrt[5]{2} + c\sqrt[5]{4} + d\sqrt[5]{8} + e\sqrt[5]{16}$, with $a$, $b$, $c$, $d$, and $e$ integral. Find $a^2 + b^2 + c^2 + d^2 + e^2$.

Solution: 5. By multiplying both sides by $\sqrt[5]{2} - 1$ and noting that the numbers $1$, $\sqrt[5]{2} = 2^{1/5}$, $\sqrt[5]{4} = 2^{2/5}$, $\sqrt[5]{8} = 2^{3/5}$, and $\sqrt[5]{16} = 2^{4/5}$ are all linearly independent over $\mathbb{Q}$, we can set up five equations for five unknowns, whose solution is $a = b = c = d = e = 1$.

4. Find the nearest integer to the sum of all $x$ where $4^x = x^4$.

Solution: 5. We immediately see two solutions, 2 and 4, and that there can be no more positive roots. There must be a negative root, however: let $f(x) = 4^x$ and $g(x) = x^4$; then $g(0) = 0$ and $f(0) = 1$, but $g$ goes off to infinity as $x \to -\infty$ and $f$ goes to 0 as $x \to \infty$. Plugging in $x = -1$, we have $f(-1) = 1/4$ and $g(x) = 1$; plugging in $x = -1/2$ we have $f(-1/2) = 1/2$ and $g(x) = 1/16$. Therefore the root is between $-1/2$ and $-1$, and the nearest integer to the sum of the roots must be 5.

5. Let $x$ be a real root of the polynomial $p(x) = x^3 - 3x + 3$. Find $x^9 + 81x^2$.

Solution: This problem has been redacted. There is no integral solution to this problem.

6. Define $f(x) = x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$ Find the smallest integral $x$ such that $f(x) \geq 50\sqrt{x}$.

Solution: 2400. Noting that $(f(x) - x)^2 = f(x)$, we can solve the quadratic equation for $f(x)$ to get that

\[ f(x) = x + \frac{1}{2} \pm \sqrt{x + \frac{1}{4}}. \]

We clearly have to take the positive root (we can notice, for example, that $f(1) > 1$). The problem therefore reduces to finding the smallest integral $x$ such that

\[ x + \frac{1}{2} + \sqrt{x + \frac{1}{4}} \geq 50\sqrt{x}. \]

It is simple to note that $x$ has to be fairly large for this to be satisfied (after trying the trivial $x = 1$). For large $x$, $\sqrt{x + \frac{1}{4}}$ is very, very close to $\sqrt{x}$, so we can rewrite this as

\[ x + \frac{1}{2} \geq 49\sqrt{x}. \]
The above is again rewritten as

\[ x^2 - 2400x + \frac{1}{4} \geq 0. \]

The smallest integer \( x \) satisfying the above is obviously 2400, and since the margin of error here is \( \frac{1}{4} \), our previous approximation is justified.

Of note is that this problem is cooked: the value \( x = 0 \) is also a valid solution. We accepted either solution.

7. Let \( f \) be a function such that \( f(x) + f(x + 1) = 2^x \) and \( f(0) = 2010 \). Find the last two digits of \( f(2010) \).

Solution: 51. We have the sequence of equations

\[
\begin{align*}
 f(x) + f(x + 1) &= 2^x \\
 f(x + 1) + f(x + 2) &= 2^{x+1} = 2 \cdot 2^x \\
 &\ldots \\
 f(x + n - 1) + f(x + n) &= 2^{x+n-1} = 2^{n-1} \cdot 2^x.
\end{align*}
\]

Adding and subtracting alternate lines, we get a telescoping sum:

\[
f(x) + (-1)^{n+1} f(x + n) = 2^x \sum_{k=0}^{n-1} 2^k (-1)^k = 2^x \sum_{k=0}^{n-1} (-2)^k = 2^x \left( \frac{1 - (-2)^n}{3} \right).
\]

Plug in \( x = 0 \) and \( n = 2010 \), so

\[
f(2010) = \frac{2^{2010} - 1}{3} + 2010.
\]

The last two digits of \( 2^{2010} \) are 24 (using Euler’s theorem with \( n = 25 \), we have \( 2^{20} = 1 \mod 25 \), so \( 2^{2000} = 1 \mod 25 \), so \( 2^{2010} = 2^{10} \mod 25 \), so \( 2^{2010} = 24 \mod 100 \)). Therefore the expression \( (2^{2010} - 1)/3 \) has last digits 41, so overall the last two digits are 51.

8. The expression \( \sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ \) is equal to \( p\sqrt{5}/2^{50} \), where \( p \) is an integer. Find \( p \).

Solution: 192. Let \( \omega \) be the root of unity \( e^{2\pi i/90} \), so we have

\[
\prod_{n=1}^{45} \sin(2n^\circ) = \prod_{n=1}^{45} \frac{\omega^n - 1}{2i\omega^{n/2}}.
\]

By the symmetry of the sine (and the fact that \( \sin(90^\circ) = 1 \)),

\[
\prod_{n=1}^{45} \sin(2n^\circ) = \prod_{n=46}^{89} \sin(2n^\circ),
\]

so

\[
\left| \prod_{n=1}^{45} \sin(2n^\circ) \right|^2 = \sum_{n=1}^{89} \left| \frac{\omega^n - 1}{2} \right|^2 = \frac{90}{289},
\]

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where we’ve used the usual geometric series sum for roots of unity. The product is clearly positive and real, so it is equal to

\[
\frac{\sqrt{45}}{244} = \frac{3\sqrt{5}}{244},
\]

implying that \( p = 3 \cdot 2^6 = 192 \).