PUMaC 2010

Algebra B Solutions

1. Let the operation $\bigstar$ be defined by $x \bigstar y = y^x - x * y$. Calculate $(3 \bigstar 4) - (4 \bigstar 3)$.

   Solution: -17. We have $(3 \bigstar 4) - (4 \bigstar 3) = (64 - 12) - (81 - 12) = -17$.

2. Let $p(x) = x^2 + x + 1$. Find the fourth smallest prime q such that $p(x)$ has a root mod q.

   Solution: 19. One can check that there are roots mod 3, 7, 13, and 19, and no others for smaller primes.

3. Write $\frac{1}{\sqrt[5]{2}-1} = a + b\sqrt[5]{2} + c\sqrt[5]{4} + d\sqrt[5]{8} + e\sqrt[5]{16}$, with a, b, c, d, and e integral. Find $a^2 + b^2 + c^2 + d^2 + e^2$.

   Solution: 5. By multiplying both sides by $\sqrt[5]{2} - 1$ and noting that the numbers 1, $\sqrt[5]{2} = 2^{1/5}$, $\sqrt[5]{4} = 2^{2/5}$, $\sqrt[5]{8} = 2^{3/5}$, and $\sqrt[5]{16} = 2^{4/5}$ are all linearly independent over $\mathbb{Q}$, we can set up five equations for five unknowns, whose solution is $a = b = c = d = e = 1$.

4. Find the nearest integer to the sum of all x where $4^x = x^4$.

   Solution: 5. We immediately see two solutions, 2 and 4, and that there can be no more positive roots. There must be a negative root, however: let $f(x) = 4^x$ and $g(x) = x^4$; then $g(0) = 0$ and $f(0) = 1$, but g goes off to infinity as $x \to -\infty$ and f goes to 0 as $x \to \infty$. Plugging in $x = -1$, we have $f(-1) = 1/4$ and $g(x) = 1$; plugging in $x = -1/2$ we have $f(-1/2) = 1/2$ and $g(x) = 1/16$. Therefore the root is between $-1/2$ and -1, and the nearest integer to the sum of the roots must be 5.

5. Let x be a real root of the polynomial $p(x) = x^3 - 3x + 3$. Find $x^9 + 81x^2$.

   Solution: This problem has been redacted. There is no integral solution to this problem.

6. Define $f(x) = x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$. Find the smallest integral x such that $f(x) \geq 50\sqrt{x}$.

   Solution: 2400. Noting that $(f(x) - x)^2 = f(x)$, we can solve the quadratic equation for $f(x)$ to get that

   $$f(x) = x + \frac{1}{2} \pm \sqrt{x + \frac{1}{4}}.$$

   We clearly have to take the positive root (we can notice, for example, that $f(1) > 1$). The problem therefore reduces to finding the smallest integral x such that

   $$x + \frac{1}{2} + \sqrt{x + \frac{1}{4}} \geq 50\sqrt{x}.$$

   It is simple to note that x has to be fairly large for this to be satisfied (after trying the trivial $x = 1$). For large x, $\sqrt{x + \frac{1}{4}}$ is very, very close to $\sqrt{x}$, so we can rewrite this as

   $$x + \frac{1}{2} \geq 49\sqrt{x}.$$

The above is again rewritten as

$$x^2 - 2400x + \frac{1}{4} \geq 0.$$

The smallest integer x satisfying the above is obviously 2400, and since the margin of error here is $\frac{1}{4}$, our previous approximation is justified.

Of note is that this problem is cooked: the value $x = 0$ is also a valid solution. We accepted either solution.

7. Let f be a function such that $f(x) + f(x+1) = 2^x$ and $f(0) = 2010$. Find the last two digits of $f(2010)$.

Solution: 51. We have the sequence of equations

$$f(x) + f(x+1) = 2^x$$
$$f(x+1) + f(x+2) = 2^{x+1} = 2 \cdot 2^x$$
$$\cdots$$
$$f(x+n-1) + f(x+n) = 2^{x+n-1} = 2^{n-1} \cdot 2^x.$$

Adding and subtracting alternate lines, we get a telescoping sum:

$$f(x) + (-1)^{n+1} f(x+n) = 2^x \sum_{k=0}^{n-1} 2^k (-1)^k = 2^x \sum_{k=0}^{n-1} (-2)^k = 2^x \left(\frac{1-(-2)^n}{3} \right).$$

Plug in $x = 0$ and $n = 2010$, so

$$f(2010) = \frac{2^{2010} - 1}{3} + 2010.$$

The last two digits of 2^{2010} are 24 (using Euler's theorem with $n = 25$, we have $2^{20} = 1 \bmod 25$, so $2^{2000} = 1 \bmod 25$, so $2^{2010} = 2^{10} \bmod 25$, so $2^{2010} = 24 \bmod 100$). Therefore the expression $(2^{2010} - 1)/3$ has last digits 41, so overall the last two digits are 51.

8. The expression $\sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ$ is equal to $p\sqrt{5}/2^{50}$, where p is an integer. Find p.

Solution: 192. Let ω be the root of unity $e^{2\pi i/90}$, so we have

$$\prod_{n=1}^{45} \sin(2n^\circ) = \sum_{n=1}^{45} \frac{\omega^n - 1}{2i\omega^{n/2}}.$$

By the symmetry of the sine (and the fact that $\sin(90^\circ) = 1$),

$$\prod_{n=1}^{45} \sin(2n^\circ) = \prod_{n=46}^{89} \sin(2n^\circ),$$

so

$$\left| \prod_{n=1}^{45} \sin(2n^\circ) \right|^2 = \sum_{n=1}^{89} \frac{|\omega^n - 1|}{2} = \frac{90}{2^{89}},$$

where we've used the usual geometric series sum for roots of unity. The product is clearly positive and real, so it is equal to

$$\frac{\sqrt{45}}{2^{44}} = \frac{3\sqrt{5}}{2^{44}},$$

implying that $p = 3 \cdot 2^6 = 192$.