PUMaC 2010

Combinatorics B Solutions

1. The Princeton University Band plays a setlist of 8 distinct songs, 3 of which are difficult to play. If the Band can't play any two difficult songs in a row, how many ways can the band play its 8 songs?

   Answer: $\boxed{14400}$

   Solution: There are $5! = 120$ ways to choose an ordering for the songs that are not difficult to play. Then the setlist is $*S*S*S*S*S*$, where S represents a song that is not difficult to play, and $*$ represents a space in the setlist that can either be left empty, or filled with one difficult song. There are $\binom{6}{3} = 20$ ways to choose 3 of these spaces for the difficult songs, and $3! = 6$ ways to choose which difficult song to put in each space. Therefore, the answer is $120 \cdot 20 \cdot 6 = \boxed{14400}$.

2. PUMaCDonalds, a newly-opened fast food restaurant, has 5 menu items. If the first 4 customers each choose one menu item at random, the probability that the 4th customer orders a previously unordered item is m/n, where m and n are relatively prime positive integers. Find $m + n$.

   Answer: $\boxed{189}$

   Solution: Number the menu items 1 through 5. Without loss of generality, assume the 4th customer orders menu item 1. Then the desired probability is the probability that each of the first 3 customers do not order menu item 1, which is $(4/5)^3 = 64/125$. The answer is $64 + 125 = \boxed{189}$.

3. Let $\underline{xyz}$ represent the three-digit number with hundreds digit x, tens digit y, and units digit z, and similarly let $\underline{yz}$ represent the two-digit number with tens digit y and units digit z. How many three-digit numbers $\underline{abc}$, none of whose digits are 0, are there such that $\underline{ab} > \underline{bc} > \underline{ca}$?

   Answer: $\boxed{120}$

   Solution: If one two-digit number is greater than another, then the tens digit of the first number must be greater than or equal to the tens digit of the second number. Therefore, if $\underline{abc}$ satisfies the given condition, then $a \geq b \geq c$. Now note that if $b = c$, then since $c \leq a$, we have $\underline{bc} \leq \underline{ca}$, a contradiction. Therefore, $a \geq b > c$. Conversely, if $a \geq b > c$, then we can easily see that $\underline{abc}$ satisfies the given condition. Therefore, the problem is equivalent to finding the number of ordered pairs (a, b, c) of integers between 1 and 9 such that $a \geq b > c$.

   If $a = b > c$, we can produce all such (a, b, c) by choosing 2 of the integers between 1 and 9, and setting a and b equal to the larger integer, and c equal to the smaller integer. If $a > b > c$, we can produce all such (a, b, c) by choosing 3 of the integers between 1 and 9, and setting a equal to the largest integer, b equal to the middle integer, and c equal to the smallest integer. Therefore the answer is $\binom{9}{2} + \binom{9}{3} = 36 + 84 = \boxed{120}$.

4. Sterling draws 6 circles on the plane, which divide the plane into regions (including the unbounded region). What is the maximum number of resulting regions?

   Answer: $\boxed{32}$

   Solution: Let $R(n)$ be the greatest number of regions that n circles can divide the plane into. We want to calculate $R(n+1)$ in terms of $R(n)$.

   Suppose we have drawn n circles on the plane, dividing the plane into r regions. Suppose we draw another circle, forming k intersection points with the existing circles. If $k = 0$, then there are no intersection points, and the resulting number of regions is $r + 1$. Otherwise, the k intersection points divide the new circle into k arcs. Each arc divides an existing region into two regions, so the resulting number of regions is $r + k$. Since the new circle intersects each other circle at most twice, we have $k \leq 2n$. By definition, $r \leq R(n)$, so the resulting number of regions is at most $R(n) + 2n$.

   To show that this maximum is attainable, we need to produce a set of $n + 1$ circles such that every pair of circles intersects twice, and every intersection point is distinct. It should be easy for the reader to convince himself or herself that there is such a set for all $n + 1 \leq 6$ by drawing circles on paper. Therefore $R(n + 1) = R(n) + 2n$ for all $n \leq 5$. We can easily see that $R(1) = 2$. Repeatedly applying the recursive equation, we obtain $R(6) = \boxed{32}$.

   For a solution that doesn't require drawing circles on paper that look like they intersect properly, we can prove the following statement:

   Claim: For all m, there exists a set of m circles such that every pair of circles intersects twice, and every intersection point is distinct.

   Proof: Take a regular m-gon, and let s be the radius of the circumcircle of the m-gon. Put m circles at the vertices of the m-gon, all with the same radius, and let the common radius be greater than s. For two circles of the same radius to intersect twice, it is sufficient for the common radius to be greater than half the distance between the centers of the circles. Since the distance between the centers of any two of these circles is at most $2s$, the diameter of the circumcircle, and since the common radius is greater than s, every pair of circles intersects twice. Now suppose for contradiction that some three circles pass through the same point P. Then P is equidistant from three distinct points on the circumcircle of the m-gon, so P is the circumcenter of the triangle formed by these three points and thus the circumcenter of the m-gon. But then the distance between P and the three points is s. Since the common radius is greater than s, P is not on any of the circles, a contradiction. Therefore, every intersection point is distinct. Therefore, such a set of circles exists for all m.

5. $3n$ people take part in a chess tournament: n girls and $2n$ boys. Each participant plays with each of the others exactly once. There were no ties and the number of games won by the girls is $7/5$ the number of games won by the boys. How many people took part in the tournament?

Answer: $\boxed{9}$

Solution: The number of games won by the girls is 7/12 the total number of games, or $\frac{7}{12}\binom{3n}{2}$. The girls must win at least $\binom{n}{2}$ games, since a girl must win any game between two girls. The girls can win at most $\binom{3n}{2} - \binom{2n}{2}$ games, since a boy must win any game between two boys. Therefore, we have

$$\binom{n}{2} \leq \frac{7}{12}\binom{3n}{2} \leq \binom{3n}{2} - \binom{2n}{2}$$
$$\frac{n(n-1)}{2} \leq \frac{7}{12} \cdot \frac{3n(3n-1)}{2} \leq \frac{3n(3n-1)}{2} - \frac{2n(2n-1)}{2}$$
$$4n(n-1) \leq 7n(3n-1) \leq 12n(3n-1) - 8n(2n-1)$$
$$4n^2 - 4n \leq 21n^2 - 7n \leq 20n^2 - 4n$$
$$0 \leq 17n^2 - 3n \leq 16n^2$$

The first part of this inequality holds for all n, but the second part implies that $n^2 \leq 3n$, so $n \leq 3$. We also know that the total number of games, $\frac{3n(3n-1)}{2}$ is divisible by 12. Then $3n(3n-1)$ is divisible by 24, so $n(3n-1)$ is divisible by 8. Since n and $3n-1$ have different parity, one of $n, 3n-1$ is divisible by 8. Therefore, $n = 3$, so that $3n = 9$.

6. A regular pentagon is drawn in the plane, along with all its diagonals. All its sides and diagonals are extended infinitely in both directions, dividing the plane into regions, some of which are unbounded. An ant starts in the center of the pentagon, and every second, the ant randomly chooses one of the edges of the region it's in, with an equal probability of choosing each edge, and crosses that edge into another region. If the ant enters an unbounded region, it explodes. After first leaving the central region of the pentagon, let x be the expected number of times the ant re-enters the central region before it explodes. Find the closest integer to $100x$.

Answer: $\boxed{200}$

Solution: Color the regions black and white like a chessboard, where the center region is white, so that no two regions sharing an edge are the same color. The ant moves alternately between black and white regions, so we can consider the ant's movement two steps at a time, essentially ignoring the black regions.

The white regions consist of the central region, five similar "edge" sections, and some unbounded regions. Let C be the expected number of times the ant re-enters the central region, starting from the central region, and let E be the expected number of times the ant re-enters the central region, starting from one of the edge regions (by symmetry, E is the same for all five edge regions). If the ant starts in the central region, there is a 1/3 probability it returns to the central region in 2 steps, otherwise it moves to an edge region. If the ant starts in an edge region, there is a 2/9 probability it moves to the center in 2 steps, a 5/9 probability it returns to an edge in 2 steps, and a 2/9 chance it explodes within the next 2 steps. Therefore,

$$
\begin{aligned}
C &= \frac{1}{3}(1+C) + \frac{2}{3}E \\
E &= \frac{2}{9}(1+C) + \frac{5}{9}E
\end{aligned}
$$

Solving yields $C = 2$, $E = 3/2$, so $x = C = 2$, and therefore the answer is $\boxed{200}$.

7. We say that a rook is "attacking" another rook on a chessboard if the two rooks are in the same row or column of the chessboard. Let n be the maximum number of rooks that can be placed on a 6×6 chessboard such that each rook is attacking at most one other. How many ways can n rooks be placed on a 6×6 chessboard such that each rook is attacking at most one other?

   Answer: $\boxed{8100}$

   Solution: Consider the following arrangement of rooks, where an R represents a rook:

R	R				
		R	R		
				R	
				R	
					R
					R

   In this arrangement, each rook attacks at most one other, so $n \geq 8$. Suppose there is such an arrangement of 9 rooks. Each row has at most 2 rooks, so there must be some 3 rows with exactly 2 rooks each. Call these 6 rooks "strong."

   Suppose there are two strong rooks in the same column of the chessboard. Then these rooks each attack both each other and the strong rooks they share a row with, a contradiction. Therefore, the strong rooks all occupy different columns, so there is a strong rook in each column. Since there are only 6 strong rooks, there must be some rook R on the chessboard that is not strong. Take the strong rook that is in the same column as R. This strong rook attacks both R and the strong rook it shares a row with, a contradiction. Therefore there is no such arrangement of 9 rooks, so $n = 8$.

   Consider arrangements of 8 rooks. Since each row has at most 2 rooks, there are exactly 2 rows with 2 rooks and 4 rows with 1 rook. Similarly, there are exactly 2 columns with 2 rooks and 4 columns with 1 rook. The rooks in the rows with 2 rooks must have their own columns, so these are the 4 columns with 1 rook. Call these 4 rooks the "row" rooks. Similarly, the rooks in the columns with 2 rooks are the rooks in the rows with 1 rook. Call these 4 rooks the "column" rooks.

Therefore we can produce an arrangement by choosing the 2 rows that are to have 2 rooks and the 2 columns that are to have 2 rooks. The row rooks are in these 2 rows and in the other 4 columns, and the column rooks are in these 2 columns and in the other 4 rows. We finish constructing the arrangement by choosing 2 of the other 4 columns to contain the upper-most row of row rooks and 2 of the other 4 rows to contain the left-most column of column rooks.

Therefore the answer is $\binom{6}{2}^2\binom{4}{2}^2 = \boxed{8100}$.

8. Matt is asked to write the numbers from 1 to 10 in order, but he forgets how to count. He writes a permutation of the numbers $\{1, 2, 3 \ldots, 10\}$ across his paper such that:

   (a) The leftmost number is 1.

   (b) The rightmost number is 10.

   (c) Exactly one number (not including 1 or 10) is less than both the number to its immediate left and the number to its immediate right.

   How many such permutations are there?

   Answer: $\boxed{1636}$

   Solution: Consider the "changes of direction" of the sequence of numbers. It must switch from decreasing to increasing exactly once by (3). By (1) and (2) it must start and end as increasing. Therefore the sequence must go from increasing to decreasing to increasing.

   Let a be the unique number that's less than both its neighbors, corresponding to the switch from decreasing to increasing, and let b be the unique number that's greater than both its neighbors, corresponding to the switch from increasing to decreasing. Then the sequence is of the form $1, \ldots, b, \ldots, a, \ldots, 10$, where $1 < a < b < 10$, and the sequence is monotonic between 1 and b, between b and a, and between a and 10.

   If we fix a and b, then the sequence is uniquely determined by the sets of numbers in each of the three dotted sections. In other words, we simply have to choose which of the three sections to place each of the remaining numbers. The numbers between 1 and a must go to the left of b, and the numbers between b and 10 must go to the right of a. The numbers between a and b can go in any of the three sections. For example, if $a = 2$, $b = 8$, and we divide the numbers between a and b into the sets $\{4, 6\}, \{3\}, \{5, 7\}$, then we obtain the unique permutation $1, 4, 6, 8, 3, 2, 5, 7, 9, 10$. Therefore the number of permutations is

$$N = \sum_{1<a<b<10} 3^{b-a-1}$$

For each $1 \leq n \leq 7$, if $b - a = n$, then $1 < a = b - n < 10 - n$, so there are $8 - n$ possible values of a, each of which uniquely determines a value of b. Therefore,

$$N = \sum_{n=1}^{7} (8-n)3^{n-1} = \sum_{n=0}^{6} (7-n)3^n$$

Multiplying the first expression above by 3, we obtain

$$3N = \sum_{n=1}^{7} (8-n)3^n$$

Subtracting, we obtain

$$\begin{aligned} 2N &= (8-7)3^7 - (7-0)3^0 + \sum_{n=1}^{6} 3^n \\ &= -7 + \sum_{n=1}^{7} 3^n \\ &= -7 + 3\sum_{n=0}^{6} 3^n \\ &= -7 + 3 \cdot \frac{3^7 - 1}{3 - 1} \\ &= 3272 \end{aligned}$$

so $N = \boxed{1636}$.