PUMaC 2010

Geometry B Solutions

1. In a polygon, every external angle is one sixth of its corresponding internal angle. How many sides does the polygon have?

   [**Answer**] 14

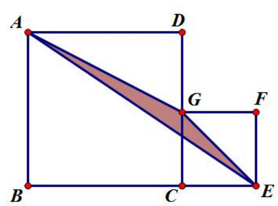
   [**Solution**] Each external angle is 180/7 degrees. So total number of sides is $360/180 * 7 = 14$.

2. On rectangular coordinates, point $A = (1, 2)$, $B = (3, 4)$. $P = (a, 0)$ is on x-axis. Suppose P is chosen such that $AP + PB$ is minimized, compute $60a$.

   [**Answer**] 100

   [**Solution**] Reflect A along x-axis to A', and connect $A'B$, the intersection point is the desired P.

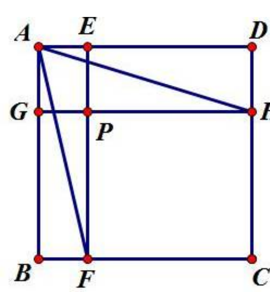
3. As in the following diagram, square $ABCD$ and square $CEFG$ are placed side by side (i.e. C is between B and E and G is between C and D). Now $CE = 14$, $AB > 14$, compute minimal area of $\triangle AEG$.

   [**Answer**] 98

   [**Solution**] [**Solution**] Connect AC, Note that two triangles AEG and CEG share same base and have equal height. So the area of $\triangle AEG$ is equal to area of $\triangle CEG = 14 \times 14/2 = 98$.

4. Square $ABCD$ is divided into four rectangles by EF and GH. EF is parallel to AB and GH parallel to BC. EF and GH meet at point P. Suppose $BF + DH = FH$, calculate the nearest integer to the maximal value of the degree of $\angle FAH$.

[Answer] 45

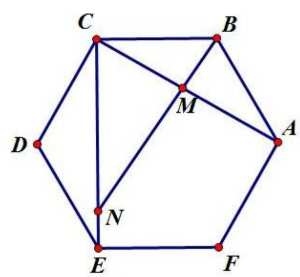
[Solution] Rotate $\triangle ABF$ counterclockwise 90 degrees to $\triangle ADM$ and we get AFH and AMH congruent (SSS congruency). Hence $\angle FAH = \frac{1}{2}\angle FAM = 45°$.

5. In a rectangular plot of land, a man walks in a very peculiar fashion. Labeling the corners $ABCD$, he starts at A and walks to C. Then, he walks to the midpoint of side AD, say A_1. Then, he walks to the midpoint of side CD say C_1, and then the midpoint of A_1D which is A_2. He continues in this fashion, indefinitely. The total length of his path if $AB = 5$ and $BC = 12$ is of the form $a + b\sqrt{c}$. Find $\frac{abc}{4}$.

   [Answer] 793

   [Solution] $AC + A_1C = 13 + \sqrt{61}$. Hence total length is $(1 + 2^{-1} + 2^{-2} + \cdots)(13 + \sqrt{61}) = 26 + 2\sqrt{61}$.

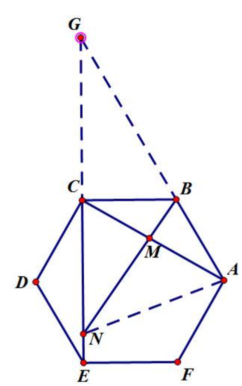
6. In regular hexagon $ABCDEF$, AC, CE are two diagonals. Points M, N are on AC, CE respectively and satisfy $AC : AM = CE : CN = r$. Suppose B, M, N are collinear, find $100r^2$.



   [Answer] 300

   [Solution] Let the side length of hexagon be 1. Extend NC and AB to intersect at some point G. Then $AB = 1$, $BG = 2$, $GC = \sqrt{3}$, let $CN = x$. We use $[XYZ]$ to denote area of triangle XYZ. Then

$$[BCG] : [BCN] = CG : CN = \sqrt{3} : x \qquad [BAN] : [BGN] = BA : BG = 1 : 2$$



Consequently,

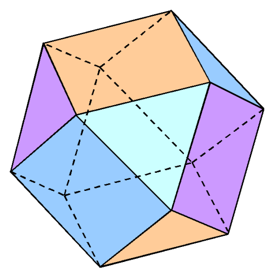
$$AM : CM = [BCN] : [BAN] = \frac{\sqrt{3}+x}{2} : x = \frac{\sqrt{3}+x}{2x}.$$

By condition, $AM : AC = CN : CE$, which translates to

$$\frac{\sqrt{3}+x}{\sqrt{3}+3x} = \frac{x}{\sqrt{3}}.$$

Solve for x, the only positive solution is 1. Hence $r = CE/CN = \sqrt{3}/1 = \sqrt{3}$.

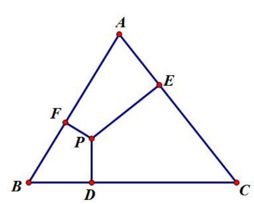
7. A cuboctahedron is a solid with 6 square faces and 8 equilateral triangle faces, with each edge adjacent to both a square and a triangle (see picture). Suppose the ratio of the volume of an octahedron to a cuboctahedron with the same side length is r. Find $100r^2$.



[Answer] 4

[Solution] A cuboctahedron is obtained by chopping off 8 corner tetrahedra of a cube. So volume of cuboctahedron of side length 1 is $(\sqrt{2})^3 - 8 \times \frac{1}{6}(1/\frac{2}{)}^3 = \frac{5}{3}\sqrt{2}$. On the other hand, volume of a regular octahedron is $1^2 \times \frac{1}{\sqrt{2}} \times \frac{1}{3} \times 2 = \frac{\sqrt{2}}{3}$. So the ratio is $1 : 5$.

8. Point P is in the interior of $\triangle ABC$. The side lengths of ABC are $AB = 7$, $BC = 8$, $CA = 9$. The three foots of perpendiculars from P to sides BC, CA, AB are D, E, F respectively. Suppose the minimal value of $\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$ can be written as $\frac{a}{b}\sqrt{c}$, where $\gcd(a, b) = 1$ and c is square free, calculate abc.



[Answer] 600

[Solution] Let S denote the area of $\triangle ABCR$. Then $S = \frac{1}{2}(BC \cdot PD + CA \cdot PE + AB \cdot FP)$. By Cauchy-Schwarz inequality,

$$(BC + CA + AB)^2 \leq \left(\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}\right)(BC \cdot PD + CA \cdot PE + AB \cdot FP),$$

hence, $\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \geq \frac{(BC + CA + AB)^2}{2S}$. So all we need to know is area of ABC, which by Heron's formula is $\sqrt{12(12-7)(12-8)(12-9)} = 12\sqrt{5}$. Therefore, minimal value is $\frac{(7+8+9)^2}{24\sqrt{5}} = \frac{24}{5}\sqrt{5}$.