Algebra A

1. [**3**] A polynomial p can be written as

$$p(x) = x^6 + 3x^5 - 3x^4 + ax^3 + bx^2 + cx + d.$$

Given that all roots of $p(x)$ are equal to either m or n where m and n are integers, compute $p(2)$.

2. [**3**] A function $S(m, n)$ satisfies the initial conditions $S(1, n) = n$, $S(m, 1) = 1$, and the recurrence $S(m, n) = S(m-1, n)S(m, n-1)$ for $m \geq 2, n \geq 2$. Find the largest integer k such that 2^k divides $S(7, 7)$.

3. [**4**] Shirley has a magical machine. If she inputs a positive even integer n, the machine will output $n/2$, but if she inputs a positive odd integer m, the machine will output $m + 3$. The machine keeps going by automatically using its output as a new input, stopping immediately before it obtains a number already processed. Shirley wants to create the longest possible output sequence possible with initial input at most 100. What number should she input?

4. [**4**] Suppose the polynomial $x^3 - x^2 + bx + c$ has real roots a, b, c. What is the square of the minimum value of abc?

5. [**5**] Let

$$f_1(x) = \frac{1}{x} \quad \text{and} \quad f_2(x) = 1 - x$$

Let H be the set of all compositions of the form $h_1 \circ h_2 \circ \ldots \circ h_k$, where each h_i is either f_1 or f_2. For all h in H, let $h^{(n)}$ denote h composed with itself n times. Find the greatest integer N such that $\pi, h(\pi), \ldots, h^{(N)}(\pi)$ are all distinct for some h in H.

6. [**6**] A sequence of real numbers $\{a_n\}_{n=1}^{\infty} (n = 1, 2, ...)$ has the following property:

$$6a_n + 5a_{n-2} = 20 + 11a_{n-1} \text{ (for } n \geq 3).$$

The first two elements are $a_1 = 0, a_2 = 1$. Find the integer closest to a_{2011}.

7. [**7**] Let $\alpha_1, \alpha_2, \ldots, \alpha_6$ be a fixed labeling of the complex roots of $x^6 - 1$. Find the number of permutations $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_6}\}$ of these roots such that if $P(\alpha_1, \ldots, \alpha_6) = 0$, then $P(\alpha_{i_1}, \ldots, \alpha_{i_6}) = 0$, where P is any polynomial with rational coefficients.

8. [**8**] Let $1, \alpha_1, \alpha_2, ..., \alpha_{10}$ be the roots of the polynomial $x^{11} - 1$. It is a fact that there exists a unique polynomial of the form $f(x) = x^{10} + c_9x^9 + \cdots + c_1x$ such that each c_i is an integer, $f(0) = f(1) = 0$, and for any $1 \leq i \leq 10$ we have $(f(\alpha_i))^2 = -11$. Find $|c_1 + 2c_2c_9 + 3c_3c_8 + 4c_4c_7 + 5c_5c_6|$.