Algebra A Solutions

1. Let $p(x) = (x-m)^k(x-n)^{6-k}$. Note that k cannot be even, as otherwise the coefficient of x^5 would be even. Hence, by symmetry, there are just two cases to check, where $k=1$ (equivalent to $k=5$) and $k=3$. For $k=1$, checking the coefficients of x^5 and x^4 respectively gives $m+5n=-3$ and

$$-3 = 5mn + 10n^2 = 5n(m+2n) = 5n(-3-3n),$$

so $5n(n+1)=1$ which certainly has no integral solutions. For $k=3$, we obtain respectively $3m+3n=-3 \implies m+n=-1$ and

$$-3 = 3m^2+3n^2+9mn = 3((m+n)^2+mn) = 3(1+mn) \implies mn=-2.$$

Hence, m and n are the roots to the quadratic $q(x) = (x-m)(x-n) = x^2-x-2 = (x-2)(x+1)$, so $\{m,n\} = \{-2,1\}$. Thus, $p(x) = (x-1)^3(x+2)^3$, so the answer is $p(2) = 1^3 \cdot 4^3 = \boxed{64}$.

2. We first define a new sequence $P(m,n)$ such that $P(m,n)$ is the largest power of 2 that divides $S(m,n)$. The relation $S(m,n) = S(m-1,n)S(m,n-1)$ implies that $P(m,n) = P(m,n-1) + P(m-1,n)$, which reminds us of the Pascal recurrence. The initial conditions become

$$P(m,1)=0,\ P(1,2n+1)=0,\ P(1,2)=1,\ P(1,4)=2, \text{ and } P(1,6)=1.$$

Since the numbers are reasonable at this point, one could potentially write out all 7^2 or so entries for $P(m,n)$, $1 \le m,n \le 7$, fairly quickly, but there are more computationally-efficient methods.

First Solution: Writing out the second row, $P(2,2) = P(2,3) = 1,\ P(2,4) = P(2,5) = 3,\ P(2,6) = P(2,7) = 4$. We take advantage of the fact that the Pascal recurrence is additive, in the sense that if $P_i(m,n) = P_i(m,n-1) + P_i(m-1,n)$ for $i = 1,2,\ldots$, then any linear combination of the P_i's also satisfies the same recurrence. Notice that $\binom{m+n-4}{m-2}, \binom{m+n-4}{m-2}, \binom{m+n-6}{m-2}$ satisfy this recurrence, and that

$$P(m,n) = \binom{m+n-4}{m-2} + 2\binom{m+n-6}{m-2} + \binom{m+n-8}{m-2}$$

for all $2 \le m,n \le 7$ based on the initial conditions. Hence, $P(7,7) = \binom{10}{5} + 2\binom{8}{5} + \binom{6}{5} = \boxed{370}$.

Note that we only use the second row because the numbers are nicer. Essentially what we are doing here is writing the grid as a sum of multiple shifted copies of Pascal's triangle. The second solution also uses the same technique, but uses shifted copies of Pascal's triangles rooted at the diagonal entries $P(m,n)$ where $m+n=9$.

Second Solution: Computing, we have $P(7,7) = P(7,6) + P(6,7) = P(7,5) + 2P(6,6) + P(7,5)$, and continuing in this manner (say by induction), we can express $P(7,7)$ as a sum of the product of elements of the 5th row of Pascal's triangle and $P(k, 9-k)$. Thus $P(7,7) = P(7,2)+5P(6,3)+10P(5,4)+10P(4,5)+5P(3,6)+P(2,7)$. We can calculate these diagonal entries $P(k, 9-k)$ fairly quickly from the inital conditions and the recurrence for P, and we obtain a final answer of

$$1 \cdot 4 + 5 \cdot 12 + 10 \cdot 16 + 10 \cdot 12 + 5 \cdot 5 + 1 \cdot 1 = \boxed{370}.$$

3. First, note that the possible end states of the machine are $\{4, 2, 1\}$ and $\{6, 3\}$, and that the machine will invariably halve itself at most every other operation, since when m is odd then the output $m + 3$ is even. Therefore, when operating in reverse order, the longest sequence will be the one that halves exactly every other time. Since the ending period $\{1, 2, 4\}$ is longer than $\{6, 3\}$ and obtains smaller values than 6, then $\{1, 2, 4\}$ end will result in the longer chain. Operating in reverse order, we can see that $\{1, 2, 4, 8, 5, 10, 7, 14, 11, 22, 19, 38, 35, 70, 67\}$ is the longest possible chain, and so the answer is $\boxed{67}$.

4. Vieta's relations give us

$$\begin{aligned} a + b + c &= 1 \\ ab + bc + ca &= b \\ abc &= -c \end{aligned}$$

From the last equation, $(ab + 1)c = 0$, so either $c = 0$ or $ab = -1$. If $c = 0$, we will have $(abc)^2 = 0$ regardless of the values of a and b. The other case is where $c \neq 0$, $ab = -1$ (in particular, $a, b \neq 0$). From the first two equations, we have $ca = b(1 - a - c) = b^2$. Because $ab = -1$, it follows that $c = \frac{b^3}{ab} = -b^3$. Hence, $1 = a + b + c = -1/b + b + -b^3$, so $b^4 - b^2 + b + 1 = 0$. We see that $b = -1$ is a root of this quartic, so factoring out $(b + 1)$, we have that

$$b^4 - b^2 + b + 1 = (b + 1)(b^3 - b^2 + 1) = 0.$$

Since $b^3 - b^2 + 1 = b^2(b - 1) + 1 < -2b^2 + 1 < 0$ for $b < -1$, it follows that $b = -1$ is the smallest possible real root of the quartic. Then $abc = b^3 \geq (-1)^3 = -1$, which is achieved for $(a, b, c) = (1, -1, 1)$, and squaring, our answer is $\boxed{1}$.

5. Observe that $f_1^{(2)}(x) = f_2^{(2)}(x) = x$. So if $h = f_{i_1} \circ \ldots \circ f_{i_k}$, then we can suppose the sequence $i_1, \ldots, i_k$ alternates between 1 and 2. If k is odd, then $i_1 = i_k$, so

$$h^{(2)}(x) = (f_{i_1} \circ \ldots \circ f_{i_k} \circ f_{i_1} \circ \ldots \circ f_{i_k})(x) = x.$$

If k is even, then either $h = (f_1 \circ f_2)^{(n)}$ or $h = (f_2 \circ f_1)^{(n)}$ for some n.

Calculating, we see that $(f_1 \circ f_2)^{(3)}(x) = (f_2 \circ f_1)^{(3)}(x) = x$. Therefore, $N \leq 2$, and indeed $N = \boxed{2}$ because if $h = f_1 \circ f_2$, then $\pi, h(\pi), h^{(2)}(\pi)$ are all distinct.

6. Rewrite the equation as $a_n - a_{n-1} = \frac{5}{6}(a_{n-1} - a_{n-2}) + \frac{10}{3}$. Define another sequence $\{b_n\}$ such that $b_n = a_{n+1} - a_n$. Thus, $b_1 = 1$ and $b_n = \frac{5}{6}b_{n-1} + \frac{10}{3}$ for $n \geq 2$, and if we define $\{c_n\}$ such

that $c_n = b_n - 20$, then $c_1 = -19$ and $c_n = \frac{5}{6}c_{n-1}$ for $n \geq 2$. Now

$$a_{2011} = a_0 + \sum_{n=1}^{2010}(a_n - a_{n-1}) = \sum_{n=1}^{2010} b_n = \sum_{n=1}^{2010}(c_n + 20) = 2010 \cdot 20 + \sum_{n=1}^{2010} c_n$$
$$= \frac{-19 \cdot \left(1 - \left(\frac{5}{6}\right)^{2010}\right)}{1 - \frac{5}{6}} + 40200 \approx \boxed{40086}.$$

7. Let $\zeta = e^{i\pi/3}$. Without loss of generality, let $\alpha_i = \zeta^i$ for each i from 1 to 6. Then we have $\alpha_3 = -1$ and $\alpha_6 = 1$. Therefore, the equations $f(\alpha_1, \ldots, \alpha_6) = \alpha_3 + 1 = 0$ and $g(\alpha_1, \ldots, \alpha_6) = \alpha_6 - 1 = 0$ show that α_3 and α_6 must be fixed by any such permutation.

   We also have that $\zeta + \zeta^5 = 1$ and $\zeta^2 + \zeta^4 = -1$. Therefore we can see that $f(\alpha_1, ..., \alpha_6) = \alpha_1 + \alpha_5 - 1 = 0$ and $g(\alpha_1, ..., \alpha_6) = \alpha_2 + \alpha_4 + 1 = 0$ are also polynomials of the desired form, so these polynomials must also be zero upon permutation, and therefore $(\alpha_2, \alpha_4) \to (\alpha_2, \alpha_4)$ or $(\alpha_2, \alpha_4) \to (\alpha_4, \alpha_2)$. Similarly, $(\alpha_1, \alpha_5) \to (\alpha_1, \alpha_5)$ or $(\alpha_1, \alpha_5) \to (\alpha_5, \alpha_1)$.

   Suppose α_2 and α_4 are fixed by a permutation that also swaps α_5 and α_1, and consider the polynomial $f(\alpha_1, ..., \alpha_6) = \alpha_1^2 - \alpha_2 = 0$. This polynomial permutes to $f(\alpha_{i_1}, ..., \alpha_{i_6}) = \alpha_5^2 - \alpha_2 = \zeta^{10} - \zeta^2 \neq 0$. Similarly, the permutation that fixes α_5 and α_1 but reverses α_2 and α_4 does not work due to the same polynomial as above. Finally, we need to show that the final two permutations do work. Clearly the identity permutation works. It remains to show that the permutation that fixes the roots ± 1 and swaps the pairs of roots (ζ, ζ^5) and (ζ^2, ζ^4) satisfies the conditions of the problem. This permutation is simply complex conjugation. Since we know that $P(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = 0$, we have

$$P(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5}, \overline{\alpha_6}) = \overline{P(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)} = 0.$$

   and thus both of these permutations work, and the answer is $\boxed{2}$.

8. The existence and uniqueness of this polynomial (up to sign) are beyond the scope of this contest, and as such we will take them for granted. Since all of the roots of $x^{11} - 1$ are powers of each other, we note that $f(x^k)$, reduced to a degree 10 polynomial by using $\alpha_i^{11} = 1$ for all i, we see that this new polynomial $f(x^k)$ must also satisfy every condition of $f(x)$. Therefore since f is unique up to sign, this new polynomial is either $-f(x)$ or $f(x)$. Since the new coefficient of x^{10} can be any of the c_j's, we know that each each of the c_j's is ± 1.

   Since $f(1) = 1 + c_9 + \cdots + c_1 = 0$, we know that 5 of the c_j's are -1, and the other 4 are $+1$. Now, look at $f(x)^2$. While this is a degree 20 polynomial, since again the only inputs we care about all have the property that $\alpha^{11} = 1$, we can restrict $f(x)^2$ to a degree 10 polynomial by simply identifying x^{11} with 1.

   Thus our reduced polynomial looks like $F(x) = B_{10}x^{10} + \cdots + B_1 x + B_0$. Note that since we know the values of $F(x)$ at all of the 11 roots of $x^{11} - 1$, by Lagrange Interpolation, $F(x)$ is uniquely determined. We can now perform the same trick we performed on $f(x)$, by replacing x with x^k for $1 \leq k \leq 10$, since all of the inputs we are interested in are powers of each other.

As before, this will shuffle the coefficients of the polynomial, and will send B_{10} to each of the other B_i's that are non-constant. Therefore $B_1 = B_2 = ... = B_{10}$. We also know that $f(1) = 0$, so $F(1) = 0$ as well, and we are given that $F(\alpha_i) = -11$. From these facts, we obtain the following equations:

$$10B_1 + B_0 = 0 \text{ and } B_1(\alpha + \alpha^2 + \cdots + \alpha^{10}) + B_0 = -11.$$

Since $1 + \alpha + \cdots + \alpha^{10} = 0$, the second equation becomes $B_0 - B_1 = -11$, and the solution to these two equations is $(B_0, B_1) = (-10, 1)$. However, the only way the constant term of $F(x)$ can be -10 is if the coefficient of x^{11} in $f(x)^2$ is -10, and the only way this can occur is if every pair of coefficients that multiplies to form an x^{11} term has opposite sign. Therefore $f(x)$ is anti-symmetric, so $c_1 = -1, c_2c_9 = -1, c_3c_8 = -1, c_4c_7 = -1, c_5c_6 = -1$, and the answer is $\boxed{15}$.